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ABSTRACT

This conference proceedings from annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) contains the following research papers: "The Interactive Nature of Cognition and Affect in the Learning of Mathematics: Two Case Studies" (T.J. Bassarear); "The Value of Concept Forming in Mathematics in Schools with Second Language Medium of Instruction" (F.H. Swart); "Conceptual Difficulties in Algebra Word Problems Having Only One Variable" (A. Ganguli); "The Process of Mathematization in Problem Solving Within a Computer Environment: A Functional Approach" (C. Kieran, A. Boileau, & M. Garansson); "Abstract Impressionist Algebra" (D. Kirshner); "Canceling Cancellation: The Role of Worked-out Examples in Unlearning a Procedural Error" (M.S. Smith & E. Silver); "Methodological Elements for the Reconstruction of an Analysis Didactics: The Case Study of Convergence" (R.M. Farfan); "Concept Image in Its Origins with Particular Reference to Taylor's Series" (R.C. Uriza); "About the Heritage in Calculus Textbooks: A Definition of Integral or the Fundamental Theorem of Calculus" (F.C. Osorio); "Software Design for Learning Mathematics" (I. Harel); "Proportional Reasoning Strategies: Results of a Teaching Experiment Using Concrete Representations" (M. Maxwell, C. Luke, J. Poholsky, L. Pattison-Gordon, S. Turner, & J. Kaput); "The Computer as an Aid to Formalizing Arithmetic Generalizations" (D.M. Peck); "The Kindergartners' Procedural Understanding of Number: An International Study" (J.C. Bergeron & N. Herscovics); "Effects of Instruction On Number Magnitude" (J. Sowder & Z. Markovits); "First Graders' Understanding of the Pre-Concepts of Number" (A. Bergeron); "The Role of Spatial Patterns in Number Development" (G.H. Wheatley & J.J. Lo); "Symbolic Representation of Addition and Subtraction Word Problems: Number Sentence Errors" (H.C. Bebout & J. Ishida); "Case Studies of Children's Understanding of the Concept of Length and Its Measure" (B. Heraud); "Relationship between Spatial Ability and Mathematics Knowledge" (D. Brown & G. Wheatley); "Reflection, Point Symmetry, and LOGO" (E. Gallou-Dumiel); "Story Editing and Diagnosis of Geometry Understanding" (G.W. Bright); "The van Hiele Model of Geometric Understanding and Geometric Misconceptions in Gifted Sixth Through Eighth Graders" (M.M. Mason); "The Understanding of Transformation Geometry Concepts of Secondary School Students in Singapore" (Y.P. Soon & J. Flake); "Proportional Reasoning in Young Adolescents: An Analysis of Strategies" (S. Larson, M. Behr, G. Harel, T. Post, & R. Lesh); "Young Children's

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North American Chapter
of the International Group
for the

Psychology of Mathematics Education

Volume 2: Plenary Lectures and Symposia

September 20–23, 1989
New Brunswick, New Jersey, U.S.A.

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EDITORS' PREFACE

The theme for the 1989 PME-NA Meeting is the description and analysis of mathematical processes. This includes approaches to observing, measuring, and understanding the processes which underlie mathematical behavior, as well as models of learning, understanding, thinking, and problem solving as they relate to the gathering and interpretation of data and the measurement of behavior.

This issue of the assessment of mathematical behavior, in our view, lies at the heart of progress in the psychology of mathematics education. If we see the study of mathematics education as a study based on experience--experience which involves classroom observation and interaction as well as individual observations, measurements, and the classification of empirical data--then we are faced with an enormous challenge. How is it possible to characterize the complex, context-dependent processes of mathematics learning in a way that does them justice? How can we *organize* our experience in order to derive useful conjectures and carefully examined conclusions, in order to improve the teaching and learning of mathematics in a systematic way?

Clearly assessment is not just a matter of surface observation, or of easily-scored mathematics tests. If it were, we would forever be bound by the inherent limitations of basic skills tests, of superficially evident classroom characteristics, of easily-determined subject variables, and so forth. The issue is whether there is some meaningful, systematic way to assess deeper "mental processes" (e.g., understandings, concepts, problem-solving processes in mathematics), non-self-evident social structures and relationships, expectations (including one's expectations of one's own performance possibilities), etc. These are constructs which may not be directly observable, but can be *inferred* from the study of behavior.

Not only do we need to find new ways of approaching such assessment, but we need to make explicit some of the assumptions underlying our present methods of assessment. In their own ways, the papers in these proceedings address this issue. Some papers do so by adopting one or another model of internal processes, explicitly or implicitly. In our view, many of the differences expressed in the invited papers and the reactions to them do not reflect different empirical observations, but different assumptions about the *kinds* of observations that are important to make.

The research reports represent a variety of interests and have been organized in this volume in the following categories:

I. Affective and Cultural Factors in Mathematics Learning
II. Algebra/Algebraic Thinking

III. Calculus
IV. Computer Environments in Mathematics Learning
V. Number Concepts
VI. Geometry, Measurement and Spatial Visualization
VII. Multiplicative Structures
VIII. Representations, Metacognition, and Problem Solving
IX. Teacher Beliefs
X. Teacher Education and Teacher Development

Each research report proposal was reviewed by three reviewers with experience in the specialty using the criteria established by PME-NA as guidelines. In cases of disagreement, the program committee members studied the comments and carefully considered the proposal. This procedure resulted in denying three proposals, and placing 15 of the abstracts in a "conditional" category. Authors were provided with the reviewers' and program committee's comments, and the decision was deferred pending the consideration of the final papers.

Of the 71 proposals submitted and reviewed, 53 were accepted, of which 46 were subsequently submitted as papers for presentation and inclusion in the Proceedings. Of the 15 abstracts that were conditionally accepted, 7 were submitted as full papers for presentation, and three of these were accepted.

The poster and video session topics also range widely. They include student errors, problem solving, computer instruction, learning styles, and teacher education and development in mathematics.

We would like to thank the other members of the Program Committee and the Local Organizing Committee for their valuable assistance, and all of the reviewers for their generously contributed time and expertise. We are especially grateful to Thomas L. Purdy, our Conference Secretary, without whose editorial assistance and organizational expertise these Proceedings would not have been possible.

Financial support has been provided in part by the Center for Mathematics, Science, and Computer Education and the Graduate School of Education at Rutgers University, and by the New Jersey Department of Higher Education.

Carolyn A. Maher
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(July 1989)

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History and Aims of PME

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the Group and the PME-NA Chapter are:

1. To promote international contacts and the exchange of scientific information on the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians, and mathematics teachers.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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AFFECTIVE AND CULTURAL FACTORS IN MATHEMATICS LEARNING

THE INTERACTIVE NATURE OF COGNITION AND AFFECT IN THE LEARNING OF MATHEMATICS: TWO CASE STUDIES

Thomas J. Bassarear
Keene State College

The number of adult Americans who have not mastered elementary mathematics is well-documented. It is hypothesized that in such persons, both cognitive and affective factors can play powerful roles in both helping and hindering their mathematical development. The paper presents results from a case study with two students in which strong interactions between cognitive and affective factors were observed.

The overall poor mathematical knowledge of American schoolchildren and adults has been well-documented. One of the many contributing factors is the relatively poor mathematical knowledge of the average elementary teacher. The study examines the mathematical development of two hardworking but mathematically weak students in a section of my Mathematics for Elementary Teachers course. The purpose of the study was to examine those cognitive and affective factors which help and hinder the student's ability to learn mathematics.

The author met each week during the semester with each student for 30 minute audiotaped interviews. The students solved selected problems out loud in a standard clinical interview format (Ginsburg, 1981). However, I made strategic teaching interventions when I observed maladaptive beliefs or attitudes or when I felt that some explication of specific problem solving strategies would be instructive. In this sense, the method here is a derivative of the Russian teaching experiment (Krutetskii, 1976). Finally, the students were specifically asked for their thoughts and insights concerning their progress several times during the semester.

This paper will describe three cases in this study in which the interaction between certain cognitive and affective factors strongly influenced the quality of the students' learning. There have been a number of studies documenting the importance of students' beliefs (for example, Schram & Wilcox, 1988). However, there have been few studies which have documented how specific beliefs have influenced, positively and negatively, the learning of the students (for example, Schoenfeld, 1985), and no other studies in which the interaction between cognitive and affective factors has been described in detail.

Beth's beliefs about the nature of mathematical problem solving

In the first interview with Beth, a common maladaptive belief emerged--that one solves mathematics problems by choosing appropriate computations. She was given the following problem: A car started on a trip from Amherst to Boston, 120 miles away. Unfortunately, the car ran out of gas one-third of the way through the last fourth of the trip. How many miles did the car travel before running out of gas? Beth solved the problem by first calculating 1/4 of 120 and then 1/3 of 30 and came up with an answer of 10. When I asked her to try to verify her answer, she responded, "10 into 120?" When I asked her why, she said, " I'm not really sure . . . 12 times 10 equals 120 . . . I wanted to put 10 into something. I wanted to check 10 somehow to see if there was a third of 30."

A similar belief emerged during the second interview. It seemed that if a problem reminded her of something from algebra, she would try to find a formula. She was given the following problem: The sum of the measures of the sides of a triangle is 35 inches. One of the sides is 4 times longer than the second side and 1 inch longer than the third side. What are the lengths of the sides? Her first attempt was to start with 4x and try to develop an equation. When that failed, she tried dividing 35 by 4, but abandoned that saying, "That's not right." When I asked her why, she responded, "It bothers me when it doesn't come out evenly."

Two weeks later, after solving some word problems using her newly developed skills (e.g., drawing diagrams, learning to use trial and error effectively, using partial results, making tables, and looking for patterns) as opposed to flailing blindly for a formula or simply groping about, Beth was able to explain her initial reluctance to use trial and error: "Part of the reason I didn't start doing this before is that I knew that I had to do a lot of figuring but I didn't really want to look into it because it was too much work."

At this point, I asked her if she avoided the trial and error strategies because they weren't "mathematical;" she had previously made several statements about "working mathematically." She replied that as time went on it wasn't that factor as much as "it just takes longer; if there was a shorter way, I would do it that way."

She was also able to state in her own words the usefulness of some of the newly developed problem solving strategies. For example, she said that setting up a chart was helpful because, "If I get lost, it gives me something to refer back to."

> I mentioned to her that I was excited by her progress. Her response was: I'm happy with it because I can actually see what is going on. I can see how come it works, which is something I never saw before. I never really understood . . . Before I was just doing what I had to, to get the answer and not really seeing what was happening. It left some doubt, but I knew there had to be more to it. So I put it on a shelf and said, "fine, that's just the way it is" . . . [Now] I'm figuring these out easier than before, cause I can see the steps and how to go about the different parts of it . . . It bothers me now if I get the right answer and don't know how I got it.

Several weeks later, Beth earned an A on the midterm. On the next interview, I asked for her attributions for her surprising development.

> Spending time with you, thinking through the problems, realizing what I am thinking as I was doing the problems. I'm able to do it not just by myself. It's kind of guided; somebody's there watching what I was doing, helping me to realize what's going on . . . It's not just going through the problem, but talking about what I did afterwards that reinforces what I did. It's motivating to actually be doing the problems. There's positive reinforcement when I get it right. Not only getting the right answer, but satisfaction in the way I thought it through.

I remarked that on a test she could not talk out loud and work in front of me. I asked her what she had learned that transferred even onto test situations.

> Being able to do step by step, double checking, and rethinking, "am I on the right track, does this make sense?" Even in my room, I'll take part of the problem, not trying to figure it out in my head like I used to do. I didn't think it could be as concrete as what it is . . . making as much sense as what it does, how to make it make sense, how to think it though and make things connect. As I did things, they connected and make sense.

She referred to her newly developed ability to verify her solutions herself and stated, "I know now how to go back to make sure, to check it in different ways."

She further elaborated:

> I'm more sure . . . Knowing that I can solve problems has made me look more into them. Before, If I came up to a problem like that,

> I'd be like, "I have no idea; I'll see how it's done in class." Now I know that there's a way to do it. I know that I can do it. I just have to find it. It's been a big motivator to try to solve the problems with you.

At the end of the semester, I asked her how her attitudes about learning and her beliefs about the best way(s) to learn math had changed over the semester. She wrote:

> I have learned how important it is for the person learning to actually understand what is going on. It is necessary for students to experiment and work with math so that they understand what they are doing completely because if they don't then later on these formulas and equations are going to have little relevance when they want to use them again.

In Beth's case, several maladaptive beliefs emerged--that one solves mathemtics problems by choosing appropriate computations, that one should try to find formulas, that problems should come out evenly, and that one should do as much as possible in one's head. A contributing factor to the first two beliefs was, in her words, "laziness." That, however, was only one small piece. She also did not have the necessary skills (e.g., making a table and using partial results) to solve these problems efficiently. Another attitudinal factor was that she did not have an experience of competence in mathematics, of mathematics making sense, and of valuing mathematics. When these feelings developed, she insisted on understanding the problems and felt personal satisfaction after solving a challenging problem.

Mindy's beliefs about showing all work

Although both students exhibited many similar maladaptive beliefs, there were differences and one in particular was striking. Beth realized during the semester that trying to do as much of the problem in her head was not very helpful. When I probed, she said that that's how you were *supposed* to do problems. On the other hand, Mindy believed that she was supposed to show *all* of her work. For example, on the Amherst-Boston problem referred to earlier, she made a number line and labeled every tenth number (i.e., 0, 10, 20, 30, etc.) When doing problems requiring computations, she would do virtually every computation longhand, for example, 60 X 4.

In my eyes, Mindy's belief was holding her back for a couple of reasons. First, her laborious computations often caused her to lose her train of thought on the larger problem. For example, she reduced one problem to having to find "how many times does 13 go into 420?" Instead of simply dividing 420 by 13, she used trial and error to determine how many times "13 goes into 420." Second, doing everything on paper gave a tedious and mechanical tone to her problem solving episodes.

One day while working on a problem, she calculated 75 x 3 mentally. I asked her how she got 225. She laughed embarrassedly and said, "I don't know." I assured her that I had asked not because she was wrong, but because I wanted to know how she got 225. She said, "I thought I'd impress you" and laughed again.

After she had solved the larger problem, I asked her about the mental math. "Is this mental computation something new? You just decided to do this on your own today?"

> M: Yah, I never do it. I didn't come in planning to do it. . . [It's] neat. All these tricks that people have about math, I never knew. I always do everything right out and like you did with the 2000, and you can just cancel out the zeros. I would do the whole problem, instead of looking at it that way.
> T: What do you think now?
> M: I think it's a lot easier and just being aware of them. With that table it's so easy . Cause I've been using tables a lot. I look at the tables and I find there's tons of different ways to find the answer. I just write it down and look around. [We had done some work with using tables to record partial results.]
> T: It's just like a whole new world.
> M: Yes it is. It's kind of fun . . . I think I'm actually understanding what I'm learning. I can't believe it.

Two weeks later, after the midterm on which she also earned an A, I asked Mindy to attribute the causes of her surprising success.

She mentioned the mental math. I asked her where her newly developed ability to do mental math came from. She replied:

> I think it's because I'm confident in this thing . . . and I'm trying to expand now, have fun with it. I'm trying to challenge myself. I've never done it before. I've had stuff [i.e., calculations] all the way around the pages . . . and all these different numbers, but I also think I'm starting to see the relationships among the numbers. For example, 32 x 4 is easy cause you have 2+2+2+2 and it's 128. Then 32 x 7 is not bad cause it's 32 x 4 + 32 x 3. I never saw the patterns before. When I started working with those tables I really liked how that worked, how you can get a pattern going and can see it.

Consistently throughout the semester, Mindy's comments about her learning were permeated with affective statements. At the end of the first interview, she said, "I was never good with word problems and when you mentioned word problems, I thought, 'Oh God, he's gonna think I'm a jerk becaaue I have a hard time doing these.'" On the third interview, she remarked that she was doing better than she had expected in the course. She added, "Somewhere it's gotta get hard. It's so easy now. There's a trick somewhere." When asked to attribute her success after the midterm, her remarks included the following statements. "One of the reasons I'm doing better is that I feel calmer." "The biggest part is that I'm not afraid of the class." "Coming in here [to the interviews] gives me a peaceful feeling." "I feel more confident." "I used to hate word problems, but when I see a word problem now, it's a challenging feeling. I don't mind doing them."

Mindy had believed that one had to show all of her work when solving problems. She had no specific recollections of having been told so, but simply thought that's how one was supposed to do math. I had occasionally done some mental math in class but had not pushed her to practice these ideas. Mindy's use of them seems to have been spurred by two affective factors--wanting to impress me and feeling confident enough in mathematics to challenge herself, to take a risk.

Beth's difficulties with percents

Shortly after the midterm exam, Beth was having trouble with understanding percents. One day in class she did not seem her usual self. She made little eye contact with me and her eyes seemed glazed during the lecture part of the class. The next day, she came to my regular office hours for help, and I was able to help her construct a better understanding of percents. Knowing that I was interested in how attitudes influence the learning process, she volunteered that during the difficult several days in which she had felt lost, her attitude had "gotten in the way." She elaborated:

> It got fixed into my head that I can't do anymore of these [percent problems]. . . Maybe if I had had a better attitude, my problem solving skills would have helped me out a little bit. I couldn't even figure out the basics of something that I needed to know.

In other words, her frustration and self-anger had virtually closed her access to the many problem solving and metacognitive skills which she had developed over the course of the semester.

Conclusions

For purposes of research and analysis we often separate affective from cognitive factors, and there are many research questions in which it is helpful to do so. However, this paper has discussed three of but many examples during the semester which support the belief that "affect and cognition are inextricably linked, and that we cannot separate the two" (McLeod, 1987, p.171). When Beth talked about how she was learning, many interactions between cognitive and affective factors were apparent. She spoke of developing certain problem solving strategies, of developing metacognitive skills to check the problem solving strategies, a sense of meaningfulness which made it more worthwhile to exert the effort, and feelings of satisfaction which positively reinforced her effort. Recall that Mindy's development of mental math was triggered by affective factors and that her attributions for success often had a strong affective tone, using words like "peaceful," "confident," and "not afraid."

Both students stressed several common factors with strong affective components which they felt played an important part in their mathematical development. First, they stressd the importance of a tenacious determination to understand. Mindy related an experience which occurred early in the semester when she was still full of doubt about her ability in which she had difficulties understanding the textbook's discussion of integers. Rather than coming to me and saying, "I can't understand this material; help me," she persevered, constructed a basic understanding and *then* came in with questions. Second, they stressed the importance of the mathematics making sense and being meaningful. Feeling this way was new for them and was an important component which fueled their effort. Finally, in describing the value of the interview experiences, both students also used affective descriptors. Beth mentioned the reassurance of "somebody there watching" while Mindy spoke of the "peaceful feeling" she felt during the interviews.

Given the growing concern over the number of adult Americans with inadequate knowledge of elementary mathematics and the growing number of college students requiring remedial mathematics, I hypothesize that, at least with this population, the interaction of cognitive and affective factors can strongly influence the mathematical development of many such students. A next step in improving our understanding could come from applying what we have learned from expert-novice research to investigate the differences between those hardworking but mathematically weak students who succeed and those who do not. An improved understanding of the cognitive and affective factors which help and hinder such students' development can open up improved teaching methods.

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THE VALUE OF CONCEPT FORMING IN MATHEMATICS IN SCHOOLS WITH SECOND LANGUAGE MEDIUM OF INSTRUCTION

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In this article an attempt was made to give a scientific analysis of the learning and teaching problem; a concept forming in mathematics. It seems evident that meaningful learning occurs only if new information is linked to existing relevant concepts. One of the main problems facing teachers in less developed communities today, is the establishment of new or unknown concepts to the learners. The difficulties experienced lie not so much in introducing the concepts, but in establishing it. A way had to be found to establish new concepts keeping in mind that the cognitive states associated with the concept do not exist, or are influenced by the differences in educational experiences, language and socio-cultural experiences. Research was done by the writer and his students, which proved that, by attaching the new concepts properly to the set of concepts of the pupil, within his concept framework, it is possible to obtain success in these educational systems, even with all the limitations that still exist.

THE BACKGROUND OF THE STUDY

The Education Department of the Republic of the Ciskei was very worried as a result of the poor standard of the mathematics in their schools. They then approached the author as Head of the Department of Didactics and Teaching Science of the University of Fort Hare to investigate the problem.

By way of a thorough investigation involving visits to schools, interviews with teachers, principals, inspectors of schools etc., it was clear that there was a number of problems present. Apart from other problems like inadequately qualified teachers, language problems, inadequate didactic approaches and media scarcity, we decided to concentrate on what seems to us the most important problem, namely inadequate conceptualization. We have ascertained that there seems to be a lack of understanding of basic concepts used in Mathematics. Even although pupils can define certain concepts, it still seems as though they cannot apply those concepts. Problems are thus experienced with the maneuverability of concepts in Mathematics.

THE OBJECTIVES OF THIS STUDY AND RESEARCH

The objectives of this study and research are:

1. to adapt the education system in mathematic teaching to the existing situation in respect of the number of unqualified teachers;

2. to use scarce manpower very economically in as far as qualified teachers' expertise will be utilised to their full capacity;

3. to serve as a morale booster for teachers and pupils;

4. to improve the quality and effectiveness of teaching of mathematics in schools;

5. to increase the learning outcome of final year pupils in mathematics;

6. to suggest strategies for in-service training programmes for unqualified teachers in mathematics and

7. to lay down principles for the development of teaching programmes on the core concepts in school syllabi for mathematics, utilizing video as a didactic aid.

THEORETICAL FRAMEWORK

In this study the maneuverability of concepts is regarded as the conscious application of an abstract concept in a new situation. In a given cognitive act structure the learner acts and applies an abstract concept from a preceding situation to a new situation. This implies that meaningful learning can only occur if new information is linked to existing relevant concepts (Gagné, 1970). The problem lies in the fact that when dealing with pupils in the didactic situation, teachers assume that the cognitive structures have already been established in early education and that relevant concepts already exist. This assumption is not always correct, because the cognitive foundations underlying knowledge and skill areas must have a firm foundation for proper conceptualization to take place (Gagné, 1970). Furthermore formative influences i.e. differences in educational experiences, mother-tongue influences and socio-cultural experiences can and do affect cognitive development (Gagné, 1985).

Conceptualization which includes the establishment of a new concept and the maneuverability of that concept is regarded as a didactic reality in this study. Apart from the problems identified in the previous paragraph quite a number of other problems are linked to the problem of conceptualization which either add to the problem or could be possible causes of the problem. The increasing volume of scientific knowledge, pupil growth rate, teachers' qualifications, pupil achievement, medium of instruction, the lack of continuity between the cultural world of the family and that of the school, financial problems and media scarcity, are some of the most important factors which were identified as problem areas in education contributing to poor conceptualization (Ausubel, 1969).

Research of the literature resulted in an investigation into the learning theories of Ausubel, Van Parreren and Gagné, to establish the role of conceptualization in meaningful learning. From these theories of learning it is inferred that the learner should be provided in advance with highly general concepts to which new concepts can be anchored; the learner's present knowledge including concepts is defined as his cognitive structure; the cognitive structure is represented in the learner's conceptual framework; the relation between new concepts and relevant items in the conceptual framework must be nonarbitrary and substantive; the learner must recognize the relationship between two concepts, a point of contact in various situations must be found in order that a new concept may be formed; a relevant learning intention must be present; examples of concepts should be given; language plays a very important role in conceptualization; questioning by the teacher asks the learner for demonstrations of concrete instances of the concept; and the learning of high level rules are dependent on the mastery of prerequisite lower level rules (McCall, 1952).

Taking research findings on conceptualization into account attention is given to some didactic strategies in the teaching of concepts. Gunning's seven skills i.e. translation, interpretation, application, extrapolation, evaluation, analysis and synthesis which are in accordance with Bloom's Taxonomy of Educational Objectives and dramatisation, are some of the didactic strategies identified for the teaching of concepts in a Third World context (Gunning, 1978).

In searching for a possible means of overcoming the problems pertaining to the maneuverability of concepts, the application of the video as a didactic aid is investigated.

Apart from the didactic principles other parameters are also identified to be applied in the production of the video-programmes. Some of the most important parameters are: structure, clarity and context, orientation, choice, planning, active meaning and evaluation. The identified parameters can be used to evaluate video-programmes and it can function as a structure for script-writing and the evaluation thereof (Lindeque, 1986).

THE HYPOTHESIS

A review of the literature and research studies (Gagné, 1985), led to the hypothesis that conceptualization in Mathematics can be improved by using the video as a didactic aid in the didactic situation.

METHODOLOGY

For the purpose of this research it was decided to experiment with a matrix class in Mathematics.

First of all the matrix mathematic syllabus was analysed into concepts. Thirty six important concepts were selected. On each of the concepts a video was made, plus minus five to seven minutes long each. Care was taken to make sure that the new concept that was to be explained, was based on the concept framework of the student and within the life experience and vocabulary of the student. These videos were accompanied by a pamphlet explaining what the student should know before explaining this concept and also a number of exercises to follow it up. The teacher should use this video only as a teaching aid and most definitely not to replace the teacher.

The subjects of this investigation were thirty eight pupils from Amabhele High School in Krwakrwa village in Ciskei. This village is situated in a rural area.

The research design is fourfold in nature:

> Measurement of the effect of the teaching aid on pupil achievement in the concepts covered by the video lesson;
>
> investigation into the biographical and socio-economic status of the pupils as well as an investigation into the effect of the video lessons on the pupils;
>
> investigation into the pupils' attitudes towards the teaching aids;
>
> investigation of the pupil's normal behaviour in a school setting.

It was decided to include the complete group of Std. 10 pupils taking Mathematics at Amabhele High School. The group was thus taken intact, "exactly as it exists, with all its inherent patterns of characteristics and behaviours" (Cates, 1985).

The sample thus taken was neither random nor stratified. Such samples are also referred to as incidental or accidental sampling (Behr, 1983).

The fact that pupils would be compared with themselves by analyzing the fluctuations in their achievements, brought a perfect match between experimental and control groups since the same pupil was part of both groups .

The subjects of this investigation included boys and girls. The teacher was provided with the video-programmes and worksheets for the pupils. The final instruction of the lesson had to be carried out in the conventional way by the teacher in the class. The video-programmes would only serve as a didactic aid in the lesson.

The dependent variable in this study is the pupil's achievement in Mathematics which is supposed to change as a result of the application of video-programmes on concepts (Cates, 1985). Finally the Wilcoxon Test is applied using the expected mark and the actual mark to determine statistically the significance level and thus the validity of the measured effect of video-programmes on pupil achievement can be proved.

Because of the fact that the subjects in this investigation would be evaluated during the examinations in September 1986, which is a normal practice in schools in the Ciskei, the Hawthorne effect could not influence the results. The Hawthorne effects implies that the observed person usually alters behaviour in order to gain the favour of the observer, resulting in incorrect observation.

The researcher also conducted a general questionnaire among the students to obtain biographical and other relevant data.

RESULTS

The pupils undergoing the experiment achieved higher marks that the expected marks, thus indicating that the video-programmes had a positive effect on pupil achievement.

CONCLUSIONS AND IMPORTANCE FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

The research showed clearly that the method of using well prepared video-programmes based on the transfer of concepts are improving the quality of mathematic teaching. It also proves that the best way to transfer concepts is by linking up with the existing concept framework of the student and to use media of communication which are understandable to the student. Therefore the existing framework of concepts of the student must be known to the teacher, so that he could link up with that framework in order to explain a new concept. If mathematic teachers follow this approach, especially where the students are studying through a second language medium, they are bound to get better results.

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ALGEBRA AND ALGEBRAIC THINKING

CONCEPTUAL DIFFICULTIES IN ALGEBRA WORD PROBLEMS HAVING ONLY ONE VARIABLE

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This paper analyzes the results of a written test and a follow-up diagnostic interview conducted with students in a developmental mathematics course in order to understand the causes of misconceptions in solving algebraic word problems. The students experienced difficulty in translating sentences of the type, "The sum of two numbers is 23. The smaller number is x. What is the larger number ?" The most common answer was y instead of 23 - x. Analysis of the solution strategies indicated that algebraic translation became abstract to the students using only one variable when more than one quantity was involved in a word problem.

Results from the National Assessment data indicated that ability to solve traditional word problems in algebra was poor for all groups of students (Carpenter et al, 1982). Application of algebra skills in solving word problems remained as a major area of difficulty for most students. Clement (1982) expressed concern about students' understanding of equations and how they are used to symbolize meanings. Clement (1982) stated that "understanding an equation in two variables appears to require an understanding of the concept of variable at a deeper level than that required for one variable equations" (p. 22). This paper dicusses an alternative view of the above statement. Students often experience severe difficulty in translating some standard algebraic word problems if required to use only one variable instead of two.

Standard algebra textbooks introduce the topic of solving linear equations in one variable early in the course, followed by applications where students are required to solve word problems using only one variable. Examples of some of the application problems are listed in Table 1. Solving systems of equations with two or three variables is introduced much later in the course. It is assumed that students need more experience in algebraic manipulation before they can solve simultaneous equations in two variables. This article documents information revealing that solving systems of linear equations simultaneously is a relatively easier task to compared to translating word problems involving two quantities with only one variable.

Table 1

Samples of Word Problems Students are Required to Solve with the Knowledge of Solving Linear Equations in One Variable.

Examples of problems[a]	Using only one variable students are expected to write	
1. The sum of two numbers is 107. One number is 5 more than the other number. Find each number.	One Number is x	Other Number is (107 - x) or (x + 5)
2. There are 75 coins consisting of nickels and dimes. If the coins are worth $5.95, how many of each are there?	Number of Nickels (or dimes)is x	Number of dimes (or Nickels) is 75 -x
3. $4,500 is invested, part at 8% and the rest at 10% simple interest. The yearly interest from the 8% investment was $90 more than that from the 10% investment. How much was invested at each rate?	$x is invested at 8% (or 10%)	$(4,500-x) is invested at 10% (or 8%)

[a] Source: Hall (1988).

The objective of this study was to explore the students' ability to translate sentences into algebraic relationships using different variables. The interest of this paper is to discuss one particular type of misconception in order to uncover the underlying thought processes.

PROCEDURE AND RESULTS

Subjects

The subjects were 51 students enrolled in two sections of an elementary algebra course and 49 students enrolled in one section of an intermediate algebra course offered at the General College, University of Minnesota. Most of the students enrolled in the intermediate algebra course had elementary algebra as a prerequisite. Elementary algebra sections were chosen arbitrarily; the intermediate algebra section was taught by the author. These students were then asked to pick up their test papers during the author's office hours, during which time five students were selected randomly from each section for a short interview. While handing in the test papers these students were asked to describe their solution processes.

Written Test Data.

Table 2 shows four items given on a 10-minute written test in the elementary algebra course. The test was administered after the students were introduced to solving linear equations in one variable. The author observed the classes taught by graduate teaching assistants for half an hour when the students were solving word problems supposedly using only one varable. At the end of the lesson the test was administered. A large number of students failed to answer the Question 1 correctly. The typical wrong answers given to the Questions 1, 2 and 4 by the students indicated that, even though they were not introduced to the concepts of two variables, most of them attempted to use two variables in algebraic translation.

Table 2
Performance of Elementary Algebra Students on Four Items in a Written Test

Test questions (Number of students= 51)	Correct Answer	% Correct	Typical Wrong Answer
1. The sum of two numbers is twenty-three. The smaller number is x. What is the larger number?	23-x	13	y
2. One number is four less than another number. Find the numbers in terms of x.	x, x - 4 or x, x + 4	22	x = y - 4
3. The sum of two numbers is forty-five. Write an algebraic expression in two varibles.	x + y = 45	98	
4. One number is three more than another number and their sum is twenty-three. Write an algebraic expression in one varible.	x + x + 3 = 23	41	x + y + 3 = 23

Table 3 shows the performance of intermediate algebra students on one word problem included in the first examination of the school term. The students completed and were tested on the first five chapters of the required algebra text book for the course. The second chapter of this text book introduced solving linear equations in one variable, followed by solving word problems using one varible. The problem listed in Table 3 is a relatively easier problem from the word problem section of Chapter 2. Solving linear equations in two variables is not introduced until chapter 9 (Hall, 1988).

Table 3

Performance of Intermediate Algebra Students on a Particular Word Problem on a Written Test . Number of students = 49

Test questions	Correct answer	% Correct
1. The sum of two integers is 11 and their difference is 35. Write an equation to describe this.	In one variable: either $x - (11 - x) = 35$ or, $(11 - x) - x = 35$;	6
	In two variables: $x + y = 11$ & $x - y = 35$	81
2. Find the integers in the above problem.	23, -12	72

The author analyzed students' solution processes and found that the students had difficulty in translating the word problem using only one variable. Most of the students used two variables to write the equation. Successful students found the right solution to the problem by the method of elimination, althogh solving systems of equations was not introduced in the course at the time of the examination. Explanations that document their understandings in solving the problem correctly needed to be explored, and 5 students selected randomly were interviwed.

Interview Tasks.

Tasks were specifically designed to focus on the conceptual part of relating two quantities by using only one variable. The students were asked to use only x in each of the problems. The tasks were to relate two quantities in terms of one variable when addition was involved and when subtraction was involved. For the limited scope of this paper details of the student responses are not reported here. Each student was interviewed individually on the average of five minutes per student. The following interview questions were asked, depending on the student's responses.

1. If one number is x what is the other number?
2. Why ?, or Can you answer in terms of x only ?
3. How are x and y related ?
4. What are you thinking? Or How did you get this answer ?

Table 4 reports the responses of three elementary algebra students . The other two students had responses similar to Student A. It was noted that the successful student (Student C) also thought in terms of two variables and reported the final result after solving one variable in terms of the other. An intermediate step was needed for that student to think in terms of x. Table 5 reports some of the responses of intermediate algebra students on question 1 of Table 3. It was noted that four out of five students thought about question 1 in terms of two variables. The students who knew how to solve linear equations answered Question 2 correctly.

Table 4

Elementary Algebra Student Responses on Interview Tasks. Question 1 of Table 2.

	Interview Questions			
	1.	2.	3.	4.
Student A	y	I don't know	Their sum is 23.	Since x is smaller it should be less than 12.
Student B	Hard to say	I don't know what x is.	*	X could be anything less than 23.
Student C	23 - x	*	*	I thought x + y = 23 and subtracted x from both sides.

* Question was skipped.

Table 5
Intermediate Algebra Student Responses on Interview Tasks . Question 1 of Table 3.

	Interview Questions			
	1.	2.	3.	4.
Student F	$x + 1$	•	•	Two integers.
Student G	I don't know	I know $x + y = 11$ but I don't know what x is.	Sum is 11 and the difference is 35.	If x is 10 y could be 1, but their difference...
Student H	writes $x + y = 11$ $x = 11 - y$	writes $x + \underline{?} = 11$ $\underline{?} = 11 - x$	•	x plus whatever is 11. So 11-x.
Student I	writes $x + y = 11$ $x - y = 35$	$x - 7/2 = 35$	x is 11y	$x = 11y$; $11y - y = 35$; $10y = 35$ $y = 35/10$; $y = 7/2$ and then plug in.
Student J	writes $x + y = 11$	$x = 11 - y$	Their sum is 11	Thinking $y - x$ is 35. So no solution.

* Question was skipped.

DISCUSSION

The students generally translated sentences into algebraic expressions by a "word order matching" approach, where the order of the key words is mapped directly into the order of symbols (Clement, 1982). When given the sum or difference of two numbers, a more concrete task was to think $x + y =$ the given sum or $x - y =$ the given difference rather than thinking of the two quantities in terms of one variable. The words larger and smaller in Question 1 in Table 2 made the problem even harder for Student A. That student wanted to split 23 in equal halves and think of x in terms of a concrete number. Clearly Student A had difficulty in relating two quantities with only one variable. In later discussion Student A said if the larger number was y, then he could understand the relation between x and y, and promptly wrote $x + y = 23$. This student treated both x and y as two fixed unknown quantities whose sum is 23 and retained the misconception about the variability of the symbol x.

Student B did not pay attention to the words smaller or larger, but exhibited some degree of understanding about the concept of variable by answering interview Question 4. In a subsequent conversation she revealed that the most confusing thing to her was that x could not be solved specifically and it could be any number less than 23. This inconsistency in thought prcesses might be associated with viewing the question as an equation rather than an algebraic expression. The typical wrong answer for Question 2 in Table 2 can perhaps be explained by the notion that many students view most algebraic expressions as equations and do the operations accordingly.

The Intermediate algebra students, most of whom preferred to use two variables in problem 1 of Table 3, mentioned the relative ease of using two variables as the reason for their preference. In a later discussion, when the students were shown the correct response in one variable, all five students thought it was too complex to follow. Not knowing from which section of the book the problem was obtained, a graduate teaching assistant majoring in mathematics used two variables to translate the problem. This leads one to wonder about the wisdom of asking students underprepared in mathematics to solve algebra word problems involving two quantities without first introducing the method of solving simultaneous linear equations. According to Wollman (1983), mathematics instruction should focus on the 'enhancement of the disposition toward coherence' (p.181). Once the conceptual difficulties are identified, appropriate instructional strategies which allow the students to understand the cohesion of the various methods can be designed. A unified view of word problems can be presented to the students if systems of equations are presented early in the course.

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PROCESSES OF MATHEMATIZATION IN ALGEBRA PROBLEM SOLVING WITHIN A COMPUTER ENVIRONMENT: A FUNCTIONAL APPROACH

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This paper describes a computer-supported, functional approach to problem representation and solution. A teaching experiment was carried out with 12 average-ability, seventh graders who participated in hourly sessions twice a week for a four-month period (once a week for part of the group). It was found that the functional approach to representing problem situations was extremely accessible to all subjects. However, for simple problems of the type $a\underline{x} + b = c$, some subjects preferred to use inverse operations rather than the function tool as a solving device; for more complex problems (e.g., the type $a\underline{x} + b = c\underline{x} + d$), all relied on the functional approach.

INTRODUCTION

Theoretical Framework

The process of problem solving can be divided into three separate phases: mathematization, solution, and verification. Mathematization is generally considered as that part of the process whereby the problem situation is translated into some mathematical model or form, such as an algebraic equation or computer program. Psychologists refer to this process as problem comprehension, a process which when further subdivided is said to include: (a) reading the problem, (b) forming a mental representation that interprets the information in the problem into objects with associated properties, (c) organizing the relations among those objects, and (d) representing the relations in some way, for example, as an equation.

The problem comprehension process provides the problem solver with the initial representation of the problem from which problem solving proceeds. At a gross level of analysis, there are two main approaches to problem comprehension. One can be called a direct-translation approach; the other, a principle-driven or schematic approach. Empirical studies have shown that, in algebra, for example, although competent students are able to use schematic relations to successfully solve word problems, students more typically have great difficulty in extracting conceptual relations from problems. Though cognitive research has been able to show that schemata are useful theoretical constructs, the findings of this body of studies, to date, have not been able to shed much light onto why students experience these difficulties or on how instruction might be geared to improve the situation.

An approach that appears more promising with regard to helping students with the mathematizing of problem situations is one that was first proposed by Clement, Lochhead, and Soloway (1980) and later elaborated on by Sfard (1987). Sfard tested sixty 16- and 18-

year-olds who were well-acquainted with the notion of function and with its formal structural definition in an attempt to find out whether these students conceived of functions operationally rather than structurally. An operational conception, according to Sfard, is one that views a function as an algorithm for computing one magnitude by means of another. A structural conception is one that views a function as a correspondence between two sets. The majority of the pupils were found to conceive of functions as a process rather than as a static construct--despite the instruction they had received. In a second phase of the Sfard study involving ninety-six 14- to 17-year-olds, students were asked to translate four simple word problems into equations and also to provide verbal prescriptions (algorithms) for calculating the solutions to similar problems. They succeeded much better with the verbal prescriptions than with the construction of equations. These findings supported the results of the study by Clement et al. which showed that students can cope with translating a word problem into an "equation" when that equation is in the form of a short computer program specifying how to calculate the value of one variable based on another. These findings also suggest a predominance of "operational" conceptions among students and consequently imply that instruction in the process of problem solving might be more successful if it were to emphasize procedural rather than structural approaches.

Objectives

The present study takes the perspective of the Clement et al. and Sfard studies. It is a three-year project that explores the use of computers in the teaching of algebra. During this first year, the research emphasis is on the development of student understanding of an approach to mathematizing problem situations. The computer is used as a tool to express a procedure for solving a problem. One of the main differences between the short computer programs used by the students of the Clement et al. study and the programs generated by our subjects is that Clement's environment used BASIC. In our study, the language is tailor-made, so that students are able to express their procedures in a kind of everyday language that admits significant naming of variables (Kieran et al., 1988).

One of the advantages of a procedural representation that is close to natural language is its relationship to the historical development of algebraic symbolism. The first evolutionary stage through which algebra passed was the period before Diophantus, a period which was characterized by the use of ordinary language descriptions for solving particular types of problems and which lacked the use of symbols or special signs to represent unknowns. The later development of a specialized symbolic language stripped away meaning from the language in which algebraic activity had been previously expressed. The cost is that symbolic language is semantically extremely weak, introducing the difficulty for the learner that, by suiting all contexts, the language appears to belong to none (Boileau, Kieran, & Garançon, 1987; Wheeler, 1989).

Thus, the principal aim of this year's study is to document the main phases of the processes of mathematization as they occur in students while they are generating procedural representations of certain problem situations, which representations are stated in a kind of everyday language. The hypotheses we have generated find their bases in ideas stated by Usiskin (1988) and Filloy and Rojano (1984). Usiskin has pointed out that

> in solving problems such as "When 3 is added to 5 times a certain number, the sum is 40," many students have difficulty moving from arithmetic to algebra. Whereas the arithmetic solution involves subtracting 3 and dividing by 5 (i.e., using inverse operations), the algebraic form $5x + 3 = 40$ involves multiplication by 5 and addition of 3 (i.e., using forward operations). That is, to set up the equation, you must think precisely the opposite of the way you would solve it using arithmetic. (p. 13, parenthetical remarks added)

Furthermore, Filloy and Rojano (1984) emphasize that problems which can be represented by equations such as $x + a = b$, $ax = b$, and $ax + b = c$ can be easily solved by arithmetic methods. They claim that a "didactical" cut occurs with problems that are representable by equations of the type $ax + b = cx + d$, for students are generally not able to solve this type of problem by arithmetic methods. However, this suggests that not only do students have to shift from thinking about inverse operations to thinking about forward operations, but also that they usually have to generate a written symbolic representation upon which they must apply algebraic methods.

We hypothesize that a functional approach to the early learning of algebraic problem solving in an environment supported by a computer can be conducive to helping students think in terms of forward operations without requiring that they use a static, equation form of representation--rather a procedural representation that has the dynamic, "operational," features which, according to Sfard and Clement et al., allows students to be more successful problem solvers.

THE STUDY

Subjects

In January 1989, we began a long-term teaching experiment in two schools, with 12 average-ability Grade 7 students who volunteered to participate in our three-year project. The four subjects from one school were "interviewed" for an hour, twice a week, in groups of two until the end of the 1988-89 school year. The remaining subjects from the other school were interviewed once a week, some in pairs and some individually, until the end of April, at which time three of these subjects continued on a twice-weekly basis.

In addition to the interviewer, an observer was present at every session. Her/his observation notes, as well as dribble files, the students' written work, and video tapes of each session served as data. The analysis of our subjects' work that is presented in this paper is based on the data gathered from January to the beginning of May.

The Environment

Because of page constraints, we describe the environment and our instructional procedures in the same section. The first session with each subject began with the presentation of a problem situation, without the actual question, for example, "Karen has a part-time job selling magazine subscriptions in her neighborhood; she is paid 20$ per week, plus a bonus of 4$ for each subscription she sells." The question, which would follow later in the same session, was, "How many subscriptions must she sell if she wants to earn 124$ in a week?" There were several reasons for separating the problem situation from the actual question. One was to prevent the students from attempting to solve the problem immediately. Since we were attempting to teach them a new way of "looking" at problems and at the procedures that can be used to solve them, we considered that, by temporarily removing the goal, they would be able to learn without being hindered by their old approaches to problem solving, that is, by their spontaneous use of inverse operations. A similar technique was used in a puzzle-problem study carried out by Sweller and Levine (1982, cited in Owen & Sweller, 1989) who found that means-ends analysis (i.e., working backward from the goal to the givens) could be prevented by giving subjects the same puzzle with the goal removed.

After the initial presentation of the problem situation, the interviewer then asked our subjects a series of questions such as, "What if Karen sells 2 subscriptions? . . . 5 subscriptions, . . . 12 subscriptions, . . . and so on?" She/he subsequently proposed that they verbalize the operations that they were carrying out, which operations were then written as follows:

12 x 4 gives 48
48 + 20 gives 68

After writing several of these sets of statements for the same problem situation, subjects drew a table which not only recorded the values they had calculated, but also reflected their discussion as to how the headings of the two columns were to be labeled (e.g., number of subscriptions, total amount). Finally, they were asked if they were able to write a general rule for their calculations in the form of a sequence of statements that a computer could carry out, such as:

number of subscriptions x 4 gives extra bonus
extra bonus + 20 gives total amount

This program was then entered into the computer; number of subscriptions was specified as the input variable and total amount as the output variable. The task, at this point, was to generate certain values for the input variable (beginning with the ones they had already tried on paper) in order that the computer calculate the corresponding values of the output variable. At this moment, the actual question that was to accompany the problem situation was presented. The aim was to continue entering different values for the input variable until the value of the output variable calculated by the computer matched the goal data of the problem question (in this case, 124$).

The second and third sessions followed the same general plan as the first, except that there was less and less intervention at each session. By the fourth session, subjects were on their own. Since the problem situation was still being presented before the actual question, it was suggested to subjects that they generate their own hypothetical question to go with the problem situation in order to provide some direction to their work. In fact, as soon as the problem situation was presented, the interviewer asked subjects what question they would like to answer that would go with the given situation. After subjects had created their "program" and had tried out some values to answer their question, the interviewer then presented the actual question.

From the ninth session onwards, subjects were presented with entire problems to be solved at once, that is, with both the situation and the question. It was believed that, by this time, the functional approach to representing problems and to solving them was solidified and could serve as a basis for working with a wider range of problems that were also more complex.

The Problems

We have classified the problems that were presented to the subjects of this study according to the types of equations that one could use to represent these problems. A sample of these problems and their types is provided in Figure 1. Note that the students of the study never saw nor worked with these equations; they were for our classification purposes only.

The first three sessions involved problems of the type $a\underline{x} + b = c$. From Sessions 4 to 8, the problem-types were expanded to include $a\underline{x} - b = c$, $b - a\underline{x} = c$, $a\underline{x} + \underline{x} = c$, $(a + \underline{x}) + \underline{x} = c$, and $a\underline{x} \cdot \underline{x} = c$. From Sessions 9 to 17, when the problem situation and question were presented together, the problem-types also included $b - (d\underline{x} + e \cdot a\underline{x}) = c$, $\underline{x} + (\underline{x} + a) + \underline{x} + (\underline{x} + a) = c$, and other variations of multiple occurrences of the variable on one side of the equation. From Session 18 onwards, problems that can be modeled by equations with occurrences of the variable on both sides of the equal sign were attempted, for example, $a\underline{x} = b + c\underline{x}$ and $a\underline{x} \pm b = c\underline{x} \pm d$.

Commercial airplanes cruise at very high altitudes. Their descent to landing is gradual. Suppose you are on a plane whose altitude is 10 000 metres. It then starts its glide to landing, dropping at a rate of 450 metres per minute. How long will it take the plane to reach an altitude of 1900 metres? [$b - a\underline{x} = c$]

The number of students in your school is 15 times the number of teachers. If the total number of teachers and students is 256, how many students and teachers are there in your school? [$\underline{x} + a\underline{x} = c$]

I have 25$ in my pocket. For the class party, I buy 7 bags of chips and 3 cases of soft drinks. One case of soft drinks costs 4 times the price of one bag of chips. If I have 4,48$ left after making these purchases, how much did each item cost? [$b - (d \cdot a\underline{x} + e\underline{x}) = c$]

The concession manager at the Montreal Forum offered two pay plans for people willing to sell peanuts in the stands for the Canadiens hockey games. The first pays 28,68$ plus 0,17$ per bag sold. The second pays 11,00$ plus 0,38$ per bag sold. For what number of bags sold will these two methods give exactly the same pay? [$a\underline{x} + b = c\underline{x} + d$]

Figure 1. Sample of problems and their type. (Our thanks to J.T. Fey and M.K. Heid for some of the word problems of our study.)

It is to be noted that for all problem-types, the functional representation of the problem as a sequence of single-operation relations was the same. As well, the first part of the solving procedure, that is, using trial values for the input variable, did not vary from one problem-type to another. The singular difference occurred in the last part of the solving procedure, with respect to the output variable: For all problems except those with occurrences of the variable on both sides of the equation, solving the problem required generating an output that matched one of the numerical data of the problem; for all problems that could be modeled by equations with several occurrences of the variable on both sides of the equation (e.g., $a\underline{x} + b = c\underline{x} + d$), solving the problem required generating two outputs that were equal.

However, since solving and verification procedures are beyond the scope of this paper (see Kieran et al., 1988, for a discussion of some of the solving methods used by other subjects in this environment), the following presentation of preliminary results focuses primarily on what our subjects did during the first phase of problem solving, that is, on the representation or mathematization phase. Solving procedures are discussed only to the extent that they interacted with representation procedures.

RESULTS

Note that the schedule that was followed in the school where the subjects were seen twice a week is the one that is reported here. In the second school of the study, each of the three main phases described below took longer to accomplish. However, the results for both schools are included.

First Three Sessions

There was considerable variation in the approaches used by subjects during these three introductory sessions of the study. Nevertheless, there were two overriding themes. One was the ease with which subjects took to a functional representation of the word problems. Representing situations such as, "For an evening's work, you are to be paid 10$ plus 0,15$ for each bag of peanuts that you sell," as generalized procedural statements, that is, as

no. of bags x 0,15 gives profit for sales
profit for sales + 10,00 gives evening's pay

posed no apparent difficulties. That subjects had already drawn a table of values using no. of bags and evening's pay as their column headings helped them considerably in deciding what to specify as input and output variables. After they had entered the above program into the computer and had tried out a few values for no. of bags, they were then given the question to go with the above problem situation, that is, "What sales are needed if you are to earn 25$." It was at this point that several different approaches emerged.

Two of the children continued to input trial values until the program produced an output of 25$. Two others mentally used inverse operations (i.e., they told us that they had tried (25 - 10) / .15) in order to more quickly arrive at a value for the input variable that would yield

25 as output. Most of the others did not see any value in using the program to solve these "easy" problems. Since the problems of the three introductory sessions were primarily of the $a\underline{x} + b = c$ type, they immediately saw that they could solve them by using their old arithmetic approach (i.e., use inverse operations--although there is some evidence to suggest that they were not necessarily aware that this is what they were doing).

Thus, the second theme for the results of these first three sessions is the predominance of attempts to use inverse operations to solve these problems, that is, to bypass the written functional representation that they had generated for the problem situation and to use their old arithmetic solving methods wherein their representation of the problem was not clearly separated from their solving methods.

Sessions 4 to 8

From the fourth to the eighth sessions, subjects were asked to verbalize their own question to go with each of the presented problem situations. Most of the problems that were generated were ones that a "function program" could answer. For example, to the situation, "The number of students in your school is 15 times the number of teachers," one pair of children suggested that "if the number of teachers was 10, they could calculate the number of students." Another pair suggested that they could find out the number of teachers and students altogether. However, one subject, who had been very strong in the use of inverse operations during the first three sessions, proposed that "if we knew the total number of persons, we could find the number of pupils and the number of teachers."

After solving their own problems, they were then presented with our question. Sometimes this required the addition of a line or two to their program; nevertheless, it posed no difficulty with respect to either extending their program or changing the name of the output variable.

Since most of the questions that the subjects proposed were input-output types of questions based on a hypothetical value for the input variable, there was no evidence of the use of inverse operations while subjects were working on their own questions--except for two subjects, one of whom was discussed above. When we presented our question afterward, the majority continued to rely on the functional representation that they had been using for their own particular question (or an extension of it) and to solve the problem by trial values for the input variable.

Session 9 Onwards

In Session 9, subjects were presented for the first time with the entire problem at once, that is, with both the problem situation and the question. The first problem of this session was an $a\underline{x} + \underline{x} = c$ type, that is, one where the use of inverse operations involving simply **c** and **a** would not lead to success: "The price of a radio is 33 times the price of a cassette. If a radio and a cassette together cost 324,70$, what is the price of each?" For half of the subjects, the spontaeous approach was to divide 324,70 by 33--even before trying to generate a functional representation of the problem. For the second problem of that session (a problem of the type $b - (d\underline{x} + e \cdot a\underline{x}) = c$), we again saw the same tendency to incorrectly use inverse operations (i.e., $(b - c) / (d + e)$) when the entire problem was presented at once.

That the problem situation and question continued to be presented together in succeeding sessions had the effect of provoking some subjects--for at least the next two or three sessions--to first try inverse operations, both for the simple problems (e.g., $b - a\underline{x} = c$), as well as for the more complex ones (e.g., $\underline{x} + a\underline{x} + b\underline{x} = c$). With the simple problems, they could succeed using their method; with the more complex ones, they were able to see that their inverse-operations approach was not leading to a correct solution. They seemed to become more aware that, for the more complex problems, they would not be successful unless they used the tools that were being made available to them. In the sessions that followed, they did not, in general, attempt to use inverse operations except for one problem in Session 16 ($\underline{x} + a\underline{x} + (\underline{x} + b) = c$) for which one subject used his calculator to figure out $(c - b) / a$ in order to try and find more quickly an appropriate trial value for the input variable.

From Session 18 onwards, many of the problems that were presented were those that can be modeled by equations with occurrences of the variable on both sides (e.g., $a\underline{x} = b + c\underline{x}$, $a\underline{x} \pm b = c\underline{x} \pm d$). There was no evidence to suggest that subjects found the generating of a functional representation for these problems any more difficult than for the preceding equation-types. In actual fact, the translation of these word problems into a set of one-operation relations is no different from the translation of simpler problems that do not contain several occurrences of the variable on both sides of the equation. For these problem-types, none of the subjects attempted to use inverse operations as a spontaneous first approach. This suggests that they were beginning to realize, in perhaps a vague way, that for certain kinds of problems, their old arithmetic methods would probably not work.

CONCLUDING REMARKS

The conclusions to be drawn from this work are tentative at best, since the study was still in progress at the time that this analysis was done. The findings from the first 22 sessions with our subjects suggest the following:

1. A functional approach to representing the relations of certain classes of problems was extremely accessible to the seventh grade students of the study, thus supporting the findings of Sfard and Clement et al. with respect to the viability of "operational" representations.
2. The technique of separating problem situation from question aided all subjects in developing a "forward-operations, functional approach" to representing problem situations; however, not all subjects considered this representation useful in solving certain types of problems (e.g., $a\underline{x} + b = c$). A minority of them preferred to use inverse operations for these problems. It was only when the problems became routinely too complex to be solved by inverse operations that these subjects began to use the functional representation.

3. The ease with which all of the subjects were able to both represent and solve problems from a fairly wide range of situations, including those that, according to Filloy and Rojano, are traditionally more difficult to represent and solve (e.g., $ax + b = cx + d$), suggests that this functional approach is an avenue worth pursuing as an entry into one of the more typical modes of representation--the algebraic equation. It would seem to be a small step to move from the forward operations of the functional representations that were used in this study to the forward operations of equation representations. In fact, the second phase of our three-year project will be focused on this very aspect.

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ABSTRACT IMPRESSIONIST ALGEBRA

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The processes of algebraic manipulation are compared to nonrational processes of abstract artistic expression.

> My painting is not literary. If I place a cow on a roof there is not a literary reference. It is a kind of logic of the illogical; a world where anything is possible. (Marc Chagall)

My purpose in this talk is to present a new way to think about algebraic thinking. As the intent is to stimulate rather than convince, I ask that you consider the explanatory possibilities of the ideas presented prior to applying the usual and necessary criteria of critical evaluation. These perspectives clash in fundamental ways with established positions in the psychology of mathematics and would be easy to dismiss reflexively rather than reflectively; a response which I ask you to forestall.

One of the most deeply and universally held belief about mathematics is that at base it is a domain of deliberate, rational intellection. Mathematical theory stands as a pinnacle achievement of human rational ingenuity, and this aspect of mathematics often informs our work as educationalists and psychologists. Perhaps as a reflection of this influence, theories of skill acquisition in algebra generally have a top-down character. Skills are understood to result from the assimilation and coordination of explicitly given propositions. It is this model of algebraic skill and its acquisition that is challenged in this paper.

The most comprehensive exposition of this view is to be found in the accumulating writings of John Anderson and collaborators (e.g. Anderson, 1983; 1986; 1987; Neves & Anderson, 1981). They describe in sufficient detail for computer simulation how fluency in a variety of domains including geometric proof and algebraic symbol manipulation comes to be achieved through initial instruction and subsequent practise. Anderson begins with the fundamental distinction between declarative knowledge and procedural knowledge, asserting "all incoming knowledge is encoded declaratively; specifically, the information is encoded as a set of facts in a semantic network" (Neves and Anderson, 1981, p. 60). His theory amounts to a detailed account of how knowledge is transformed from declarative to procedural form as skills are consolidated.

While Anderson's theory provides the most comprehensive statement, virtually all theories in the psychology of algebra subscribe to the same basic premise. For instance Matz (1980) proposes that *extrapolation techniques* are applied to a base of explicitly given declarative rules in creating the actual procedural rules of algebra:

> [T]he knowledge presumed to precede a new problem, usually takes the form of a rule a student has extracted from a prototype or gotten directly from a textbook. For the most part these are basic rules (such as the distributive law, the cancellation rule, the procedure for solving factorable polynomials using the zero product principle) that form the core of the conventional textbook content of algebra. These are referred to as the *base rules*. (p. 95)

The top-down view of algebraic knowledge is particularly evident in research which conceptualizes algebraic manipulation as problem solving. For instance Wagner, Rachlin and Jensen (1984) present skill acquisition as a rational process resting on "rote memorization of formulas and algorithms" (p. 7):

> A basic premise of this study was that the learning of algebra, beyond the level of rote memorization of formulas and algorithms, can be regarded as a kind of problem-solving process. That, even the application of formulas to "routine" textbook exercises involves some degree of problem-solving activity on the part of most students, at least initially. (p. 7)

These examples sample the widespread belief that algebraic skill is acquired through rational assimilation of the explicitly given rules of the curriculum, and hence that doing algebra can be considered a rationally-based activity.

A NONRATIONALIST ANALYSIS

Error analyses have been prominent in the development and evaluation of theoretical frameworks in algebra, it being reasoned that the processes of learning are most easily apprehended at their points of breakdown. Fadia Nasser, completing a masters degree at Tel Aviv University (Nasser, in preparation) described to me her attempts to classify and explain the following errors which she observed in the work of Israeli secondary school mathematics students:

$x^y \cdot x^z = (x \cdot x)^{y \cdot z}$ $\quad$ $x^y \cdot x^z = (x + x)^{y \cdot z}$ $\quad$ $x^y \cdot x^z = (x \cdot x)^{y+z}$ $\quad$ $x^y \cdot x^z = (x + x)^{y+z}$

Traditional perspectives must hold that such errors are a result of misappropriating, miscoordinating or misunderstanding explicitly given rule such as $x^y \cdot x^z = x^{y+z}$. Below is a different kind of analysis of the first of these errors in which the building blocks of cognition are taken to be not explicitly given rules but syntactic relations of a very general and universal sort. I provide no empirical support for the details of this analysis, the intention being to impart the "flavor" of processes to be more rigorously studied elsewhere.

Following is a possible psychological derivation of the $x^y \cdot x^z = (x \cdot x)^{y \cdot z}$ error:

$$x^y \cdot x^z \xrightarrow{parse} (x^y) \cdot (x^z) \xrightarrow{associative} [(x^y) \cdot x]^z \xrightarrow{movement} [x \cdot (x^y)]^z \xrightarrow{assoc} [(x \cdot x)^y]^z \xrightarrow{visual.assoc} [(x \cdot x)]^{(yz)} \xrightarrow{mult} [(x \cdot x)]^{(y \cdot z)} \xrightarrow{parse} (x \cdot x)^{y \cdot z}$$

Technical Notes

1. The step marked *movement* could have been described as commutative. Movement, however, has the exponent actually jumping from one "x" to the next; the recurrence of "x" being incidental. If movement is not itself a basic relation, a detailed analysis might look something like this:

$$x^y \cdot z \xrightarrow{parse} (x^y) \cdot z \xrightarrow{commutative} z \cdot (x^y) \xrightarrow{assoc} (z \cdot x)^y \xrightarrow{commut} (x \cdot z)^y \xrightarrow{assoc} x \cdot (z^y) \xrightarrow{parse} x \cdot z^y$$

2. The step marked *visual.assoc* differs from ordinary associativity in that it operates on concrete visual entities rather than (possibly) on abstract symbolic entities. For instance if we link horizontal juxtaposition with some abstract entity, *multiplication*, and diagonal juxtaposition with abstract entity, *exponentiation*, then ordinary associativity can have an abstract character, [*a multiplication b*] *exponentiation c* → *a multiplication* [*b exponentiation c*], as well as a visual character, [*a horizontal b*] *diagonal c* → *a horizontal* [*b diagonal c*]. But visual.assoc does not work as true associativity at the abstract symbolic level, [*a exponentiation b*] *exponentiation c* → *a exponentiation* [*b multiplication c*], only at the visual level, [*a diagonal b*] *horizontal c* → *a diagonal* [*b horizontal c*]. This raises critical questions about whether the syntax of algebra is encoded in visual or abstract terms. (See Kirshner, 1989, for a related discussion.)

DISCUSSION

Examining the applications of "associativity," "commutativity," "distributivity" and (perhaps) "movement," above, shows them to be much broader that the formal mathematical rules normally associated with these terms. The perspective advanced here is that such broad syntactic relations are not derived from mathematical experience, but are fundamental and very general elements of human cognition. Perhaps such syntactic relations are tied to natural language grammars which also employ associative, commutative and distributive structure. Perhaps they underlie the syntactic structure which artist Marc Chagall (above) identifies as the "logic of the illogical."

It is the image of elementary algebra as "symbol play" that I most want to convey in this paper. Algebraic manipulation in its synchronic as well as its diachronic contexts can be regarded as part of a natural syntactic outpouring of the human psyche. Such a perspective requires careful dissociation of the contemplative character of mathematical theory from the aesthetic character of notational form which William Rowan Hamilton (1837) reminds us are distinct motive forces for engagement in mathematics:

> The Study of Algebra may be pursued in three very different schools, the Practical, the Philological, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the *agere*, the *fari* or the *sapere*,) is eminently prized and sought for. (p. 293)

I propose that attaining and expressing algebraic skill is principally a linguistic rather than an intellectual exercise.

To sketch out this scene in more detail, it is not only errorful performance which is to be conceived as resulting from creative syntactic invention, but competent performance as well. For instance, application of the $x^y \cdot x^z = x^{y+z}$ rule can be regarded as having a similar derivation to that presented above; perhaps embodying some element of distributivity in the collecting together of occurrences of "x". The point, however, is not simply that the rules of elementary algebra are acquired non-rationally. I propose further that the processes of doing algebra remain fluid, converging eventually towards stable rule structures which need not correspond with the rules which happen to fill the algebra textbooks, and subsequently our introspective reports about our algebraic knowledge. (See Kirshner, 1987a, Ch. 5; and Kirshner, 1987b, for examples of introspectively inaccessible rule structures.) In other words, the logic of doing elementary algebra, like the logic of doing abstract art, is implicit (though formal mathematical theories provide interesting reconstructions of the former).

I have no evidence for the specific details of the $x^y \cdot x^z = (x \cdot x)^{y \cdot z}$ derivation presented above, but some broader empirical considerations suggest that some such syntactic explanation is likely to be correct. If, as postulated in traditional approaches, algebra learning rests upon rational assimilation of explicitly given rules, then errors ought to reflect the rational and conceptual confusions of the novice. But the errors documented by Nasser (above) like the errors generally acknowledged to be endemic to algebra learning have a clearly syntactic rather than semantic character. Surely it is futile to attempt to predict with any reasonable degree of specificity the presence of the $x^y \cdot x^z = (x \cdot x)^{y \cdot z}$ error on the basis of semantic and conceptual categories.

If error analysis paradigms are to be used honestly and to full effect then errors must be viewed as perturbations of competent performance. It follows, then, that competent performance, like errorful performance, is syntactically grounded. I suggest that the consistent opting for rational analyses of algebraic skills in the psychology of mathematics reflects pretheoretic commitments which need to be reassessed.

Doing mathematics can be likened to playing a game of jump rope. Rope skipping as a social activity is regulated by rules; rules to make sure that everyone gets a turn, to determine who has won, etc. But we rarely lose sight of the fact that the purpose of skipping is to exult in coordinated motion; not to follow rules. In the case of mathematics, perhaps the functional imperatives of academic pursuit tempt us to forget that a primary motive force of mathematical activity is exultation in the native human capacity for syntactic expression, and that reasoning serves crucial but anciliary functions of organizing and structuring mathematical expression and of delimiting it from other modes of syntactic expression such as abstract art.

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CANCELING CANCELLATION: THE ROLE OF WORKED-OUT EXAMPLES IN UNLEARNING A PROCEDURAL ERROR

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This study examined the effectiveness of using worked examples to remediate a procedural error in the domain of algebra. Extrapolating from research on learning from worked examples, it was hypothesized that exposure to instructional treatments involving the use of correctly worked examples or incorrectly worked examples, in which the cancellation error was made salient, would help students eliminate that procedural error from their repertoire. Both treatments were successful in helping students significantly reduce the number of cancellation errors made when simplifying rational expressions. Evidence suggests that worked-out examples may be useful in helping students detect and correct procedural errors.

Procedural errors are nearly ubiquitous phenomena of interest to researchers studying mathematics learning and performance. For many researchers, the challenge has been to organize and classify the errors made in a specific domain and to understand the origins and consequences of the erroneous procedures (e.g., Brown & Burton, 1978; Matz, 1982). It has been argued that procedural errors often occur when students lack linkages between conceptual and procedural knowledge (Hiebert & Wearne, 1986) or have only partial, incomplete linkages (Silver, 1986). VanLehn (1986) contends that procedural errors result from insufficient and/or ambiguous examples that cause students to overgeneralize. According to Matz, procedural errors in the domain of algebra are often the result of "reasonable, although unsuccessful, attempts to adapt previously acquired knowledge to a new situation" (1982, p.25).

Matz identified three error categories based on the presumed origin or source of the error: extrapolation errors, errors reflecting impoverished knowledge, and execution errors. Matz's third category contains errors that are of a purely procedural nature. By contrast, the first two categories contain errors that appear to result from disconnections between conceptual knowledge and the set of symbolic referents. For example, Matz asserts that some errors — called extrapolation errors in her scheme — are generated when a student, confronted with a problem that bears some surface similarity to a problem for which a correct procedural rule is known, incorrectly applies the old rule to the new situation. She argues that these errors are conceptually based and result from inadequate knowledge of semantic constraints.

One particular error that Matz identified as an extrapolation error is the well-known cancellation error, in which an expression like $\frac{x}{2x + y}$ is incorrectly equated to $\frac{1}{2 + y}$.

According to Matz's analysis, this error is generated by the incorrect generalization of a correct simplification rule that equates $\frac{ax}{a}$ to $\frac{x}{1}$ or x. According to Matz's analysis, this error results from an inadequate knowledge of the semantic constraints implicit in the correct rule, such as requiring that the numerator and denominator have a common factor.

There are at least two important pedagogical issue related to errors. One issue is how to help students learn without errors. In a study of the use of worked examples as a substitute for equation solving in learning algebra, Sweller and Copper (1985) found that students who had been presented with worked examples made fewer errors on test problems of the same type than students who were given conventional problems in which they generated their own solutions. Sweller and Cooper contend that this improved performance is due to the fact that acquisition of expert-like procedural and equation-solving schemas is enhanced by the study of worked-out examples, in which students can more easily and directly process the relationships between and among the initial state, the goal state, and the intermediate steps needed to achieve the goal. In contrast, students who solve conventional exercises focus their attention on goal attainment and pay less attention to relationships among solution states, thereby inhibiting the development of powerful schemas. In closely related work, Zhu and Simon (1987) compared the performance of students who learned through conventional instruction to solve quadratic equations by factoring and students who learned by studying a carefully constructed set that combined worked-out examples and conventional exercises. Zhu and Simon reported that the students who studied worked-out examples were at least as successful on all performance measures than the students who had learned by conventional methods. Moreover, evidence obtained from interview protocols indicated that the students who studied the worked-out examples did not simply learn rote procedures but rather learned with understanding. According to Zhu and Simon, this successful learning from examples is due, at least in part, to the fact that the students who studied the worked-out examples were actively engaged in their learning — spending their time studying the examples and examining relationships among solution steps — rather than passively listening to a teacher's explanation.

A second pedagogical issue related to procedural errors is how to help students *unlearn* errorful procedures that have become part of their repertoire. One approach to unlearning involves increasing students' ability to detect their errors. In a study of error-detection processes in statistical problem solving, Allwood (1984) suggested that error detection may be initiated when a student makes a match between a specific error stored in memory and an occurrence of the error. He also noted that errors were more salient to good problem solvers (those who had the fewest errors in their final solutions); therefore, they detected a larger portion of both their conceptual and procedural errors and were more suspicious of problems in which specific errors might occur. Incorporating error-detection activities into mathematics instruction may help make certain errors more salient and help students to recognize and correct errors that are part of their equation-solving repertoire.

The purpose of the study presented here was to investigate the unlearning of one specific algebra procedural error — cancellation — and to compare two treatments designed to help students detect and correct the error when simplifying rational expressions. One treatment involved the use of correctly worked examples, and the other treatment involved using incorrectly worked examples, in which the cancellation error was made salient. Of particular interest was the relative effectiveness of these treatments in helping students eliminate the cancellation error from their equation-solving behavior.

METHOD

Subjects

The subjects were 18 college students (10 female and 8 male) enrolled in an elementary algebra course offered through a College of General Studies which attracts non-traditional students. The students ranged in age from 18 to 55 years and had diverse mathematics backgrounds including some having no prior experience in algebra, some having had a lack of success in their first algebra encounter, and others exhibiting high levels of mathematics anxiety.

Design

Testing occurred near the end of the term in two sessions that were separated by one week. During the first session, subjects completed a paper-and-pencil pretest containing five problems involving cancellation (e.g., simplify $\frac{4 + \sqrt{17}}{4}$). This type of problem was chosen because the subjects in the study had just completed a section on solving quadratic equations and many students made cancellation errors in the final steps of solving quadratic equations. Following the first session, the pretests were corrected and each subject was randomly assigned to one of the two treatment groups.

The second session began with the returning of the corrected pretest and administration of either Treatment 1 (T1) or Treatment 2 (T2). Treatment 1 (Correctly Worked Examples) consisted of five worked-out examples, in which rational expressions, similar to those which had appeared on the pretest, had been simplified correctly. The instructions stated that all problems were solved correctly and that students should study the procedure used for each exercise. Treatment 2 (Incorrectly Worked Examples) used the same five rational expressions as T1; however, in T2 the examples were simplified incorrectly and the cancellation error was made (e.g., $\frac{8+\sqrt{5}}{4} = \frac{\overset{2}{\cancel{8}}+\sqrt{5}}{\underset{1}{\cancel{4}}} = 2+\sqrt{5}$, in which the first term of the numerator and the denominator are divided by a common factor without taking the second term of the numerator into consideration). The instructions stated that each problem had an error which should be identified and explained. Students in each treatment group were given ten minutes to complete the activity, after which they were given the posttest, which was identical to the pretest.

RESULTS

Table 1 reports the mean success and error rates by treatment group for the pretest and posttest. The data clearly indicate that each treatment was successful. Students in each treatment group solved significantly more problems and made significantly fewer errors on the posttest than on the pretest. Although there were many cancellation errors on the pretest, students in each treatment group made essentially no cancellation errors on the posttest. There was no significant difference between the two treatments in their effects on students' performance.

Table 1 Mean Production of Correct Solutions and Errors on Pretest and Posttest

Treatment	Session	Mean Number Correct	Mean Number of Errors	Mean Number of Cancellation Errors
T1	Pretest	2.8	2.2	2.1
(N=9)	Posttest	4.8	0.2	0.1
T2	Pretest	2.0	3.0	2.4
(N=9)	Posttest	4.7	0.3	0.0

Combining results for the two groups, the mean score on the posttest was 4.7 (out of 5) while the mean pretest score was 2.4 (out of 5). Of the 18 subjects, 16 produced error-free posttests (8 from each of the two treatment groups) while only five subjects produced error-free pretests. Overall success on the problems increased from 43 correct responses on the pretest to 85 correct responses on the posttest. The subjects produced a total of 47 errors on the pretest, 41 of which were classified as cancellation errors, whereas the posttest yielded five errors, only one of which was classified as a cancellation error.

The treatments were equally successful, but not equally easy to use. Students assigned to T1 had no difficulty using the treatment sheet, but some students had difficulty using T2. In particular, some students had questions about why the problems were incorrect and sought explanations from the instructor or other students to clarify their understanding. Perusal of the treatment sheets for T2 students revealed that (a) some students gave a written explanation of the error but others did not, and (b) some students reworked the T2 problems but others did not. Despite some confusion in using T2, students were apparently able to benefit greatly from the treatment since they exhibited almost perfect performance on the posttest — a significant improvement over their pretest results.

Although it was not required, or even suggested by the instructor, most students decided to 'redo' the problems they had done incorrectly on the pretest. About one-half of the students who had made errors on the pretest corrected their errors before taking the posttest; an additional one-quarter corrected at least some of the errors they had made on the pretest. The proportion of students who exhibited this behavior was the same for the two treatments.

DISCUSSION

Sweller and Cooper (1985) argued that having high school students study correctly worked-out examples instead of solving conventional exercises facilitated the acquision of desirable equation-solving schemas and enhanced the development of expertise. In this study, the use of worked-out examples has also been shown to be effective when used with more mature subjects in a remediation setting. Moreover, the findings of this study also suggest that students may be able to learn effectively from incorrectly worked-out examples, in which a particular error is made salient. The success of treatment 2 may be due to an increased sensitivity to errors that lead to a triggering of an error detection and correction process.

Given the simplicity and short duration of the treatments, the strong positive results are quite surprising. Both treatments were extremely effective in helping students eliminate their cancellation errors. The dramatic success of the treatments stands somewhat in opposition to Matz's contention that the cancellation error is due to a lack of conceptual knowledge, since it is unlikely that these very brief treatments would have effectively corrected such a deficiency. Perhaps instead, for the population in this study, the cancellation mistakes represented execution errors caused by not carefully monitoring the procedure. If so, then it is reasonable that exposure to correctly or incorrectly worked-out examples helped the students to be somewhat more thoughtful or more aware of the existence of the cancellation error and its consequences.

Why should these brief, simple treatments have been so effective? Related work on learning from correctly worked examples (e.g., Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Zhu & Simon, 1988) has suggested that students learn successfully from worked examples if they actively process the information presented in the examples — probing connections between steps and solution states and also between information presented in the task and information stored in long-term memory. Although the treatments were brief, it is likely that subjects were actively engaged in learning from the treatment sheets. Further research is needed to examine the behavior of subjects as they study either correctly or incorrectly worked-out examples. Further research is also needed to study the long-term consequences of these treatments. Examination of subjects' behavior on their final course examination suggested that the treatments may have had positive long-term effects, but more convincing evidence is needed. Have students effectively eliminated the error or does the error lie submerged waiting to reappear in another context? If these treatments were used in earlier algebra instruction, at other points in the curriculum when cancellation errors commonly occur, would the student be able to avoid the error in the future when faced with problems that traditionally invoke the cancellation error?

The availability of the pretest may have acted as an impetus for students to relate the information on the treatment sheets to their own behavior in simplifying rational expressions. Many subjects corrected at least some of their pretest errors after studying the treatment sheet. Since the posttest was identical to the pretest, the correction of pretest errors during the treatment is likely to have contributed to the increased success on the posttest. Nevertheless, it is unlikely that the total, dramatic improvement for **all** subjects, including some who did not correct their pretest during treatment, could be due simply to this unintended aspect of the treatments. Given the extent of the success achieved, it is more reasonable to assume that subjects were able to learn effectively from the treatment and incorporate this learning into their behavior when simplifying rational expressions on the posttest. Moreover, even if correction of the pretest proves to be a significant factor in the improved performance, the dramatic impact still needs to be explained. Further research is needed to clarify the basis for the success of these treatments.

Although the use of worked examples is fairly common at the early stages of mathematics instruction aimed at teaching correct procedures, worked-out examples are usually abandoned in favor of conventional problems after students have some familiarity with the procedures involved. The study reported here suggests that correctly or incorrectly worked-out examples may be used successfully to help students detect, avoid and correct their procedural errors. Studies concerning the efficacious use of worked-out examples in teaching procedural skills, or other mathematical topics, may provide important information for reformers of mathematics curriculum and instruction who wish to increase the amount of class time available for higher-level mathematical thinking, reasoning and problem solving.

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CALCULUS

METHODOLOGICAL ELEMENTS FOR THE RECONSTRUCTION OF AN ANALYSIS DIDACTICS: THE CASE STUDY OF CONVERGENCE

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The present discussion is the first part of a research aiming to incorporate heuristic processes and reasonings in the reconstruction of educational mathematics discourse, specifically as related to the concept convergence of infinite series and its association with the notion of stability in fluid systems. We present the characterisation of phenomenology intrinsic to this concept, as well as that of the constructs which conform such phenomenology. This methodological approach permitted us to establish the relationship between the stationary state and the study of convergence of infinite series.

Some Opening Considerations

The starting point in our research is to consider the epistemological perspective in Mathematics education research [8,23], and the need to incorporate therein the study of didactic phenomenology, in order to enrich educational discourse [10]. In its first stage, which we report here, the purpose of the study was to characterize the intrinsic phenomenology of the concept of infinite series convergence, both numeric and functions series, i.e., we wished to find the phenomena which characterise the concept itself, in its historical genesis. These phenomena have been buried under a process of didactical transposition that nowadays prevents us to perceive the essential meanings which permitted its construction [5].

When analysing original sources, we perceive the presence of constructs inherent to the formation of the concept, i.e., all the scaffolding which the subject builds up when acting on the object, in order to gain access to the concept. In this process, and in which a heuristic procedure is unavoidable [5]. Thus, we attempt to reconstruct the educational mathematical contents, in the light of the basic elements which formed the theory, and which are now absent from the textbooks. Our motivation stems from the practice in the National Program for Mathematics Teachers and Training (*) (University level), and from research reports [17,21,22] where difficulties to gain access to Calculus concepts have been discussed.

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The approach to which we have resorted consists of detecting the "intellectual abilities" and "reasoning processes" present in the construction of mathematical theory. Our sources have been, the original texts [9,14,15,19] and specialised treatises dealing with the history of mathematics in the 19th century [11,12,20]. We have not ignored textbooks of the 19th century [2,14,18] nor those that are currently used in our school system, such as [3].

Our approach, while detecting constructions inherent to the concept at the time it originated, establishes in a natural way, an environment where mathematical objects acquire meanings, and this sets a pattern for the reconstruction of educational discourse. This redesign must take into account certain variables, such as the fact that the epoch's cultural context was different from ours and, therefore, a straightforward transference is not possible. Hence, the task of "adapting" to our time must necessarily pass through an experimental phase. At the present time, we are in the process of designing an experimental setting which will constitute the second part of our research; nevertheless, we have obtained partial results with the mathematics teachers involved in the training Program who teach in engineering schools (not yet reported). Such results point to the feasibility of the methodological approach we have chosen.

Characterization of Intrinsic Phenomenology

Work with series - although not the study of their convergence - is considered from the Middle Ages, a time when the first efforts were made to establish the scientific bases for the study of variability and change phenomena. Thus, we find in [16] the statement and solution of a variability problem where geometry is shown to be handled as a "useful tool" [6]. The concrete referent in which the problem arises is strictly physical, a constant characteristic throughout the development of Calculus. Another major trait of work with series which occurs from the 14th, to the 18th centuries, is the computation of the sum in question, except perhaps for Taylor's series [1], which becomes a "prediction" instrument. A remarkable example of sum computation is given by Euler, who uses a very peculiar style of mathematical discovery in the 18th century, called "eulerian induction" by Abel. Euler's solution is based on the establishment of analogies [6]. In this conceptual stage, "Computation of sums of infinite numerical series", an epistemological obstacle, made itself apparent, and prevented advancing to the next phase, namely, the fact of attributing to the sum the same nature as that of the terms in the series. The roots for this are to be found in the concept of function as an arbitrary analytic expression which prevailed in the 18th century.

One might think, based on the examples given above, that proceeding heuristically is a characteristic of the times when answers are sought to concrete problems. But this is not so. In the realm of Calculus itself, theory construction recourse is made to this "procedural style"; thus we find that the first list of convergence criteria for series in modern terms, given by Cauchy [4] derives from the comparison with series of which their sum is known [7].

For Cauchy to undertake the task of giving convergence criteria, it was first necessary to recognize that a study of convergence was required. This came to pass with J. Fourier's work (1822) on heat transference. In this work, an equation governing the behaviour of this phenomenon is deduced by the use of the "parametric prediction" instrument which was natural at that time [1]. What is significant in that work for our purposes, however, is the treatment of the problem following the establishment of the equation which consists in finding the **PERMANENT STATE** the temperatures will eventually reach, without suffering further changes with time.

The solution to the problem is an infinite trigonometric series whose coefficients must be determined. Since the series represents a system of temperatures, and since these cannot be infinite, the convergence of such series is established. In order to prove this assertion, Fourier uses several resources ranging from "eulerian induction" to transforming the solution into an integral - showing that this tends to a constant - and going through several particular cases [13]. Through-out this development there is a presence of the concrete physical referent which allowed him, in spite of "false" considerations - according to present day mathematical knowledge - to start the study of convergence. And even more, since the stationary or permanent state is a single one, so is the solution, thus showing not only the convergence of the series but also the unicity of the differential equation solution. In short, finding the stationary state necessarily leads to the verification of convergence and, therefore, to its study; thus viewed, the physical problem and the mathematical concept are *indistinguishable*. Therefore, consideration of the stationary state marks an epistemic break; the problem of computing infinite series sums is transferred to the study of their convergence.

At present, both phenomena are equivalent in textbooks; and yet, in the conceptual terrain they are not. This conceptual stage, "the study of convergence," gives rise to a new difficulty related to the concept of uniform convergence, which Fourier called "fast" as distinguished from the "slow" one, or non-uniform, into which we will not go here.

In sum, when determining the phenomenology of the concept under study, one observes that physical contexts are necessary to establish two conceptually different stages, and also that for each of these we detect the presence of the constructs used to conform the concept, as well as the epistemological obstacles which prevent us to proceed to the next stage. At the risk of being too schematic, we present the following comprehensive view, which embodies everything mentioned above.

Intrinsic Phenomenology of the Convergence Concept in Its Inception.

Stages	Concrete Referent	Constructs	Epistemological Obstacles
•Computing sums of infinite numerical series	R_1	s_1 s_2 s_3	•Identifying the nature of the sum with the terms in the series (for functions).
----------epistemic break----------			
•The study of series convergence	R_2	c_1 c_2 c_3 c_4	• "Fast" and "slow" convergence.

R_1: The study of variability and change.
R_2: Determining the STATIONARY STATE.

s_1: Completing areas
s_2: Algebraic induction
s_3: Analogies from the finite to the infinite case.

c_1: Numerical induction.
c_2: Consideration of particular cases.
c_3: Analogies from the finite to the infinite case.
c_4: "Comparison lemma."

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CONCEPT IMAGE IN ITS ORIGINS WITH PARTICULAR REFERENCE TO TAYLOR'S SERIES

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In the development of research focused on the use of physical thought in the construction of mathematical concepts and processes specifically related to the notion of prediction in physics and to the appearance of the analiticity notion in mathematics, we have characterized the concept images underlying the Taylor's series concept, which in various contexts and at different times have been established as paradigms. This became possible after the establishment of concept intrinsic phenomenology. The present work constitutes the epistemological study of the research.

One of the research lines which has recently shown a vast fecundity in the didactics of sciences, is the one focusing on the study of misconceptions, and it has shown the existence of conceptual schemes which are essentially unalterable by knowledge taught at school. This has heralded the need for a correct interpretation of the epistemic frameworks wherein such knowledge unfolds and it has made it possible to point out its importance in those aspects inherent to the nature of concepts and processes, as well as its possible phenomenological contextualisation [6]. In this direction, we have sought the concept images of a Taylor's series in its origins and have pointed out some clues for a reconstruction of its nature. The selection of the concept images we report here was done after consulting and interpreting three kinds of references, namely, the usual treatises containing historical and epistemological discussions [7-10]; a few old textbooks renowned for its influence in education (covering from late 17th century to early 20th century) [11-14]; and, of course, original writings [15-32]. The choice of this theme obeys to the recognition of its repeated presence as a driving idea in a vast diversity of conceptual constructs in the beginnings of mathematical physics.

CONCEPT IMAGES

Next, we present various paradigmatic schemes which are related to Taylor's series in several contexts and historic moments, and which we classify in eight models:

§ 1. **Binomial Regularity Model.** This concept image features the perception and use of regularity in binomial developments. It focuses on numbers

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and variable quantities, and even though it does not specify the latter's variations, it shows their operative similarity with numbers, within this model it is proposed to build several numerical tables for simple and reliable computations, and also to extend the results of early 17th Century handling of decimal numbers, to the notion of variable quantity. Characteristic results include Pascal's Triangle, Newton's Binomial, and Newton-Gregory's Interpolation Formula. Taylor's series appears in all of them, even if it is not recognized as the organizing pattern [17-19,21,22].

§ 2. Variable-Variation Model. It consists in recognizing and systematically using the idea that the part contains the information of the whole; i.e., whereas the variation of some variable quantities with respect to others--physical or geometrical--is studied, it is recognized that instantaneous or local variation provides integral information on the phenomenon. With this model, a mathematical description is sought of the laws of planetary movement, and for the movement of rigid bodies and ideal fluids on Earth; also sought is the construction of a concept-algorithmical instrument which studies variation and change in nature. Let us say that the context in which this model arises is characterized by the answers it gives to questions posed under the light of the Galilean break with Aristotelian physics. And also by the accumulation of a great variety of concrete problems where the drawing of tangents and curve quadrature are recognized as inverse problems. Notable results here include the appearance of an algorithmic Calculus, the coining of terms for concepts such as flexion, fluent, force, acceleration, differential, and integral, and the fact that the Series comes to be identified with the phenomenic principle of the Fundamental Theorem of Calculus [14-16,19-21].

§ 3. Parametric Prediction Model. It refers to the determination of the future state (the neighboring state, more widely speaking) by means of the information on the current state (the **de facto state**, more generally). It already studies variables as eventually quantifiable objects. In other words, once the significant parameters of variables and of their successive variations have been determined, the future (neighboring) state is predicted. This model does not necessarily evoke temporal variables; it proposes, for instance, to determine the temperature within a certain region, when the value at its border is known. Another result with this model would be a method of proposing differential equations. Taylor's series is recognized as the adequate instrument for prediction, and it and each one of its terms are imbued with such meanings. In this approach, one observes the use of the notion of orders of magnitude associated with the series, and a reorganization program in the light of the emerging notion of analicity [14,21,26].

§ **4. Parametric Evolution Model.** This model rests on the determination of those laws governing the system behavior, provided that the initial state is known. It studies variables associated to physical descriptions of movement of rigid bodies, particles, ideal fluids, heat, electricity and magnetism, and generally, the mechanics of continuous media. In this search for the establishment of a rational mechanics, a method outlined in Models 2 and 3 becomes more firm; namely, that in the determination of comprehensive laws, it becomes necessary to resort to the study of infinitesimal elements. Its results include the theory of Differential Equations and the initial stages of Complex Analysis [12,15,19,23-31].

5. Polynomial Approximation Model. This scheme comes very close to our practice on the Series and it features the reduction of function computation to the computation of polynomials. To do this, a succession of them is constructed in such a way that it converges to the function in question, and that they inherit the point behavior of the function; a margin of error is estimated. Results in this field would include the method for the construction of such polynomials, interpolating polynomials and the beginning systematic studies of convergence. This approach, strongly inspired by the resolution of equations by approximative methods, and uses the Series as an approximation instrument, and its remainder as the error [11-13,27].

§ **6. Functional Metamorphosis Model.** This procedure evokes the fact of transforming a function into an infinite polynomial expression, while the notion of function is permeated by that of an arbitrary analytical formula. In this scheme, series are developed by means of the binomial or through the systematic application of Taylor's Series and in it the study of convergence becomes more precise. It focuses attention in the Remainder, and it is used for the resolution of differential equation problems which are not solved by means of elementary functions [11-14,24,27].

§ **7. Inductive Generalization Model.** This model comes into being after the recognition of a set of previous results, by using the limit as an organizing concept, the mean-value theorem for derivatives, and the study of "arbitrary functions" and their classification in Classes. In it, the Series becomes one more result of theory, and the function it played in the previous models is here reversed [11-13,24,27].

§ **8. Complex Analicity Model.** Its roots can be located in the recognition of derivability in Complex Analysis (a distinction is made from real-valued functions). Approaches are designed to fundament such an area, based only in power series, and the analytic continuity property is recognized as belonging to this approach, as are numerous results on the analytical functions where the Series is used iteratively [11,12,27,30].

FINAL CONSIDERATIONS

Briefly, we shall say that we have located the presence of a stratification, by periods, in concept images, when analyzing the origins of the Taylor's Series concept; in it, we have recognized a life of its own, inherent meanings and specific contextualisations, as well as uses, in the development of the theory.

By taking stock of general elements, we shall say, broadly speaking, that 17th century concept images are characterized by two traits: first, one centered in the recognition of the Series images in the development of algorithms and numerical-algebraic patterns; and a second one, by the systematicity of point studies of rigid body movement phenomena, as well as of curves. The latter is an idea which should bear fruits in future Calculus development. In the 18th Century they featured their presence in the algorithmic consolidation of Calculus, and the manipulation of series, whereby novel discoveries were obtained, problems with function given by formulas were approached, as well as others using some special functions; and this permitted the integration (in the 19th Century already) of the general theory of real variable and of complex variable functions. The original source for all this history continued to be mathematical physics, whose horizons during the 17th and 18th Centuries was limited to the mechanics of particles, rigid bodies, and fluids, lake, water or air, by applying Newtonian physics to continuous media mechanics. In the 19th Century, this perspective extended its boundaries when heat, elasticity, electricity and magnetism, and the electromagnetic theory of wave propagation, were accepted as (fluid) subjects of study. This interaction focusing on differential and integral equations, and in variation Calculus, generated many new concepts and new formulations of problems, new integral formulas and series expansions. Taylor's Series, therefore, preserves its role as "developer" of new results [2-4].

These approaches to research allow the perception of knowledge-building patterns, through their concept images. Just recognizing them is already one result, for the action possibilities deriving from them are obvious when we contrast them with those of modern didactics. The existence of such situations has allowed the enrichment of didactic communication and the organization of experimental observations within the framework of the options provided by this approach.

In terms of didactic models for the teaching of Calculus, this research shows two essential approximations coexisting in current didactics: one stemming from the works of Newton, Euler, and Laplace, among others, where the expression of the Series carries with it a meaning pertaining to physical sciences, and which turns it into a natural construct for a vast diversity of problems; and the other, proposed by Cauchy's work on analysis, where the Series becomes one more result of theory, a consequence of the limit concept and of the mean value theorems. As we know well, the latter scheme is the one present in current Calculus courses, and the other one, although used in various contexts, is absent from the topics transmitted in College and University level teaching.

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ABOUT THE HERITAGE IN CALCULUS TEXTBOOKS: A DEFINITION OF INTEGRAL OR THE FUNDAMENTAL THEOREM OF CALCULUS?

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Our topic here is the teaching of Calculus at the intermediate and university levels within our National Education System. The framework of the present research has been limited, basically, to the characteristics of mathematical discourse in teaching, particularly as it refers to Integral Calculus; and also the "measurement" of its conceptual system, as a result of the teaching practice.

One of the major elements we consider in the development of our research is the mathematical discourse in teaching. It is precisely its characteristics that we examine, considering them an unavoidable material basis at any discussion level involving the teaching of mathematics; in our case, this refers to Integral Calculus.

We pose the problem in the following terms: if we view the teaching of Calculus as a problem of knowledge transference [2] and we assume that its main components are teachers, Calculus concepts, and students, then what we wish to analyze is how the mathematical discourse "behaves" throughout the path: teachers - concept - student. And then, if no change makes itself apparent during this passage, we attempt a Redesign of the Didactic Discourse of Calculus (**RDDC**). Within our educational system, the teacher-student relationship is such that the former cannot make the knowledge of Calculus transmissible, whereas the latter does not succeed in learning it; and this situation has become stable.

One possibility in this approach, which we have chosen, is to study the behavior of Calculus discourse during the transit, hoping that its product, the **RDDC**, actually provokes a change or otherwise breaks this stability which is detrimental for the teaching process.

Now, then, what is the **RDDC?** To begin with, the characteristics of the present discourse have fabricated one single model of Calculus knowledge transmission, thus provoking a paradigmatic situation. It is here, we believe, where the parts involving a didactic problem are to be found. Secondly, development of the **RDDC** must stem from the practical discourse; its aim should be to understand, and to make others understand, a mathematical content.

Considering this, then the **RDDC** must be viewed, in a certain sense, as governed by the patterns of pure mathematics, but its standards should not become a "mathematical obstacle." In other words, a theory develops when we propose a certain theme; "obstacles" arise incidentally, and are not necessarily a factor in fostering development. It being so, the **RDDC** has its own objective, i.e., to make others understand Calculus, together with the establishment of its own pattern of mathematical rigorism and its own mathematical tools.

Let us consider one case: the didactics of Integral Calculus (**IC**). We study the behavior, in the didactic discourse, of the break with the older concept of antidifferentiation, due to the use of the new concept, the limit of a sum as justified--in modern-day textbooks--by a good development of integration theory. And we consider the links that might exist between the break and the preceding elements which made possible the construction of such theory.

The Cauchy-Riemann definition of integration leads to a **sui generis** set of mathematical instruments: the partition of intervals, upper and lower sums, the convergence of a sum. This set of tools we have called the Riemann Apparatus (**RA**).

It should be pointed out that the treatment of the RA in this discussion is based on a pragmatic aspect: the teaching of integration; this, however, is neither to deny the historical importance of the **RA** construct in mathematics, nor the change both qualitative and quantitative--the Calculus of functions, rather than that of variable quantities--which contributed to the development of Integration Theory.

The situation, in the field of teaching, is as follows. A constant element in the study of students' behavior faced with the **RA** is that the few who succeed in applying the integration methods do not succeed in understanding the integration process. Moreover, the **RA** propitiates a new theoretical body which, **a priori,** is not linked to the previous one, i.e., Differential Calculus.

One appreciation of a didactic nature is that the Cauchy-Riemann definition of integration is prematurely presented in the **IC** discourse and is even, perhaps, unnecessary **[1].**

The notion preceding **RA** is antidifferentiation, which constitutes an equivalent model, in mathematical contents, to the **IC** discourse which at present appears in Calculus textbooks.

In this model, integration appears as a relative concept, i.e., related to a given differential: "A **quantity which through differentiation yields a proposed**

differential is called the integral of such a differential" [3]. Here, the very concept of integral is linked to the differential of a variable or a quantity, a situation which leads to the unification of differential and integral Calculus is one theoretical body.

The existence model of an integral quantity is a property which is intrinsic to the integration concept. In order to install ourselves in an integration problem, we must necessarily start from a differential, which is derived from a preconceived quantity. And then, solving the integral means to find such a preconceived quantity. This existence model can be represented by the following scheme:

$$X \to dX \Rightarrow \int dX \to X$$

Within this context, if a "Fundamental Theorem of Calculus" is required in the discourse, it would assume this form: *"if* $dX = dY$, *then* $X\text{-}Y = C$ *(a constant)."* Since in **IC** the variation quantity would be studied in order to find the primitive quantity, in this sense the "Theorem" would merely mean that if two variable quantities admit, with respect to a parameter, equal infinitesimal increments dx and dy, such quantities must remain constantly equal to each other, or must always differ by the same quantity; hence $X - Y = C$.

Such being the case, the topic of the discourse would be, precisely, the differential of a preconceived quantity. In my opinion, then, it would become natural to think of problems, both geometrical and mechanical, in terms of infinitesimal variation.

If we were to calculate, for instance, a preconceived area in terms of its variation with respect to a parameter, $dA \to A(x)$. $A(x)$ would be the area quantity varying with respect to x. Once the area quantity has been found, the next problem is to establish a general strategy to compute its numerical value with respect to boundaries a and b of parameter x; $dA \to A(b) - A(a)$. By the "Fundamental Theorem of Calculus," the area quantity must be expressed as $A(x)+C$ *where* C *is a constant.* Thus, the strategy consists in setting a point of reference, let us say **M**, which precedes the lower boundary of parameter x, a in this case $(a < M)$. And once thus assumed an initial area value in M, the area value from M to a will be given by $A(a) + C$, and the area value from M to b, by $A(b) + C$. Finally, since what we wish is to compute is the area value from a to b, this will be given by $A(b) - A(a)$. All the above wording can be reduced to the expression:

$$\int_a^b dA = A(b) - A(a)$$

At this point, one is led to think of the relationship that might exist between this concept of integration and that of quadrature.

With this advantage of hindsight, to abandon the **RA** in the **IC** discourse would seem, **a priori,** to part with the geometrical intuition, i.e., quadrature. It is quite true that a natural attitude for the computation of non-quadrated regions is to quadrate them. Now, to compute an area starting from its known differential, dA, does not entirely lead us away from this natural attitude. In the first place, an area differential, dA, inherits the nature of area A, and it is this dA we quadrate. Such a dA is, so to speak, a small piece of A, as little as one wishes, in such a manner that dA may be "considered" as the area of a small rectangle ydx, where y is its height and dx its base.

Thus, both the limit of a sum, and "taking" a differential element require, in a certain sense, the notion of quadrature.

Finally, I would like to stress the fact that, in our approach to the study of the old **IC** didactic discourse **[r, 3]**, the driving idea in the model of integration theory is the taking of a differential element. As I said above, this idea preceded the **RA**, and what is more, it prevailed, in didactic discourse, in the work on integration by Cauchy **[4]**. Meanwhile, in the development of Mechanics this idea prevails unrivalled, and is even used as a methodological element seeking to give a better explanation of its fundamental laws. It can be said that the pattern which accounts for the formulation of the Integration Theory is precisely, the taking of a differential elements.

Some final considerations

The scheme that governs Integration Theory ix $X \rightarrow dX \rightarrow \int dX \rightarrow X$. Extrapolating, this is equivalent to the integration of derivatives, a concept which is in agreement with the fundamental problem of the General Theory of Integration. It is in this sense that a mathematical model can be built, whose definition of integration can be $\int_a^b f = F(b) - F(a)$, where F is a primitive function of f, equivalent to Lebesgue's integration.

Thus, the scheme would possess the same conceptual domain as the limit of a sum in the General Theory of Integration **[5]**.

Summing up, I would like to use Foucault's words to emphasize the relevance and orientation of this research in the didactics of mathematics: **"...it will not be a question of knowledge described in its progress towards an objectivity wherein, at least, our present science can be recognized; what we shall try to bring under the light will be the epistemological realm, the episteme where knowledge, considered outside any criterion which refers to its rational value or its objective**

forms, sinks its positivity and thus show a history which is not that of its growing perfection, but that of its conditions of possibility" [6].

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COMPUTER ENVIRONMENTS IN MATHEMATICS LEARNING

SOFTWARE DESIGN FOR LEARNING MATHEMATICS

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This paper describes a four-month-long experiment in computer-based learning and mathematics education called Instructional Software Design (ISD) and presents the project's strategies and philosophy, as well as some aspects of the evaluation. The participant ISD class, comprised of fourth-grade children at a Boston public school, learned about programming and fractions while developing software in LogoWriter. The research was quantitative and comparative as well as qualitative. Two control classes were selected: one class had equivalent exposure to Logo, but no exposure to ISD; the other used Logo only once a week in a "computer laboratory." All three classes followed the regular mathematics curriculum, including a two-month unit on fractions which coincided with the ISD project. Pre- and post-tests were administered to the experimental and control groups; in addition, the experimental class was carefully observed and interviewed throughout. The case studies and the evaluation revealed greater mastery of Logo and fractions by the experimental class than for either control class and greater acquisition of metacognitive skills. In the course of discussion, the ISD approach of using programming as a tool for reformulating mathematical knowledge about fractions and Logo, is compared to other approaches to learning Logo, notably teaching Logo *per se* in isolation from a content domain.

THE INSTRUCTIONAL SOFTWARE DESIGN PROJECT (ISD)

One of the main ideas behind this research is the creation of a constructivist learning environment (Papert, 1980, 1986, 1988) that resembles a design studio or a professional software company, and where the computers are used as a medium for children's learning mathematics through building representations and explaining their knowledge to others.

At Project Headlight, there is no long hall way leading into one classroom called the "Computer Lab" where children are being walked into once or twice a week for a 45-minute "Computer Literacy Class" or "Computer Programming Class." Rather, there are two large open areas (the Pods) housing four large circles of 100 computers, and each pod is surrounded by 6 classrooms with no doors. At Headlight, children use computers at least one hour a day, for working on their different computer projects, as an integral part of their regular homeroom activities. One of these pods with its many computers, turned into a software-design studio during the ISD project (Harel, 1988, 1989). This was the environment for the seventeen fourth-grade children who were given the opportunity to construct a personally designed piece of instructional software that would explain something about fractions to some intended audience.

In ISD, fourth-grade children worked with great intensity and involvement, over a period of four months (close to 80 hours in total), on a subject that more often elicits groans or yawns than excitement--namely fractions. What made fractions so interesting to these children was that they could work with them in an environment and a context that mobilized creativity, personal knowledge, and a sense of doing something more important than just getting a correct answer. Each child had complete freedom to choose a particular topic within the general area of fractions, as well as the freedom to choose how to teach about it, what screens to design, whether to use graphics or text or both, and so on. These students, through the use of their computers, tackled complex mathematical problems and representations, they worked on large-scale, meaningful projects, had great reflective responsibility for their own learning, and were able to work in a variety of styles whose differences reflect gender, ethnicity, or individual personality .

The ISD environment required the deep involvement of all the participants. ISD included interactions and reciprocal relations among the children, teacher, researcher, members of the MIT staff, and sometimes visitors--all of whom worked at their personal computers, walked around the computer-area, talked together, helped each other, expressed their feelings on various subjects and issues, brainstormed together, or worked on different programming projects individually and collaboratively. Knowledge of Logo programming, design, and mathematics was communicated by those involved; and the children, much like the adults in this area, could walk around and observe the various computer screens created by their peers, or look and compare the different plans, designs, and representations for fractions and algorithms in their Designer's Notebooks.

In this noisy, flexible, creative, and productive software design studio young children were developing mathematical knowledge and ideas. They were learning with no workbooks or worksheets, but with a different kind of a structure. They became instructional software designers, and were representing knowledge, building models, and teaching concepts on their computer screens. They were thinking about their own thinking and other people's thinking--simultaneously--as means for their own learning.

There are several reasons why the computer is an outstanding medium for learning through instructional designing, as well as for investigating children's processes of designing, producing, and representing. "Instructional software designing and programming" meant the building of a system that has an instructional purpose and the format of an interactive lesson. In this context, the instructional systems were constructed by children, over a long period of time, and were composed of many computer procedures or routines (i.e., isolated units) that were connected to each other for the purpose of teaching or explaining fractions to other children. Furthermore, unlike most computer routines or programs, instructional software is a collection of programs designed while seriously considering the human interface. The instructional software must facilitate the learning of something by someone--a real person.

Creating instructional software on the computer requires more than merely programming it, more than merely presenting content in static pictures or written words, more than managing technical matters. When composing lessons on the computer, the designer combines knowledge of the computer, knowledge of programming, knowledge of computer programs and routines, knowledge of the content, knowledge of communication, human interface, and instructional design. The communication between the software producers and their medium is dynamic. It is a constant planning and replanning, representing, building and rebuilding, blending, reflecting, reorganizing, evaluating and modifying. Software designers must constantly work back and forth between the whole lesson to its parts, between the overall piece and its sub-sections and individual screens. Because of the computer's branching capabilities, the designer has to consider multiple routes each user might take, and the non-linear relationship between the lesson's parts can grow very complex. Moreover, while using the computer, the producer needs to design interactions between the learner and the computer: designing questions, anticipating users' responses, and providing explanations and feedback. The child-producer who wants to design a lesson on the computer must learn about the content, become a tutor, a lesson designer, a pedagogical decision-maker, an evaluator, a graphic artist, and so on.

The psychology of instructional software designers is different from that of learners in a regular classroom. Instructional software design is a complex, active, and time-consuming enterprise. It requires that software designers invest a large amount of time in learning to program, create, and implement their *own* ideas and explanations about the subject matter involved. They do not "sit and listen," but are personally involved in their learning/teaching enterprise, and take pride in it. They are the ones who make it happen. Perkins (1986) says, tthat in "knowledge as design" environment he problem's meaning is not given by the problem itself; rather, the designer imposes his own meaning and defines his own goals before and during the process. The goals and the sub-goals may change over that period of time, and keeping track of these changes is a central interest when the design task is not for the purpose of "getting it right," but is aimed at learning and developing thinking skills. Schon (1987) also analyzes how different designers (e.g., architects) impose their own meaning on a given open-ended problem, and how they overcome constraints (created by themselves, or given as part of the problem they solve) and take advantage of unexpected outcomes. In the process of learning, and when educational practices are at issue, many of the research questions change. Even the rare existing literature on the processes of software design or software engineering in no way attempts to investigate what could designers learn through the process of software design, or how the designer's content knowledge develops through the process of software design.

ISD FOR LEARNING LOGO PROGRAMMING

Many researchers over the last decade stated that young children find it difficult to learn Logo in the first place, or to pursue a richer route into programming and other metacognitive and meta-learning skills of various kinds (i.e., other more general thinking skills such as planning, note-taking, explaining, representing, or reflecting). Researchers in the field, basing themselves on these limited findings and many other educational practice constraints have sought "better" instructional techniques and more sophisticated teaching methods for Logo programming *per se*, hoping that the development of "better Logo Courses" would result in the learning of programming and its transfer (e.g., Carver, 1986; Perkins et al. 1985, 1987; or articles in Pea and Sheingold, Eds., 1987).

However, some of the reasons for many of the pessimistic findings in the research on children's learning and understanding of Logo programming, or on children's programming and their cognitive development, were partially and possibly related to several limitations in the studies themselves, and not necessarily to the children's cognition, learning, and problem-solving abilities with Logo. Piaget, for example (whose cognitive theories strongly influenced the creation of Logo and its educational philosophy) argues that a child's knowledge of something results from his own progressive constructions in wide and meaningful contexts, and that each time one prematurely teaches a child something which he could have discovered and constructed for himself, the child is prevented from inventing it and therefore from understanding it completely. With this quite radical and rather "Logoish" perspective in mind, we suggest that one limitation of the previous studies might be the fact that the children did not program intensively or extensively, and were not given the chance to explore and experience Logo in a wide variety of contexts over long periods of time. Other related limitations of previous studies might be their involvement with programming for the sake of programming, their not providing children with meaningful contexts and tasks for programming, and their failure to integrate the programming process into the learning of other subjects.

When we describe Debbie, for example (see Harel, 1988, 1989), we emphsized that the important thing was *what* she was programming. The justification for taking the trouble to learn to program would not be only the cognitive gains of learning to program, nor anything as believing that she would need to know Logo when she grows up, but what she could do with this skill in the *here and now*. But in fact there is no need to justify taking the trouble to learn to program, for learning programming was the same activity as using the program to express herself about such matters as fractions.

Programming in the ISD context also meant that children were able to take control over the computer, and to use it as a very personal medium for learning other content knowledge. It is the most powerful and general way to use a computer. That is why languages like Logo (Papert, 1980) and Boxer (diSessa, 1989) will continue to be important in education; because they give students a great control of computers, putting that resource at their disposal in service of student-oriented projects and meaningful activities, because powerful applications like HyperCard, many word processors, and more advanced databases have programming options just below the surface, programming concepts are increasingly indistinguishable from these tools (Papert, 1988).

ISD FOR LEARNING ABOUT RATIONAL NUMBERS AND THEIR REPRESENTATIONS

Unlike whole numbers, which children largely come to grasp informally and intuitively out of school, learning the rational-number system is confined almost ecxlusively to school. Fractions figure prominently in the curriculum each year from the second grade on. Even so, several national assessments of children's mathematical achievements have found that children's performance on fraction-ordering and computation, for example, was low and accompanied by little understanding (Tierney, 1987; Post et al., 1985).

Fractions are ideal tools for learning about number systems and representational systems in mathematics. The understanding of the rational-number representational system is a privileged piece of knowledge among the other pieces of rational-number knowledge. Representations form part of the deep structure of rational-number knowledge, wheras algorithms are the surface structure (e.g., Janvier, 1987). One of the goals of ISD was to involve children in exploring and learning the system of representations of fractions intensively and then assess their knowledge of basic rational-number concepts and algorithms.

There is a diversity of knowledge about rational numbers, including 1) the subconstructs (e.g., ratio, part-whole, operators, fractions, decimals, or percentages), and 2) the representations for each subconstruct and for the whole rational-number system (e.g., words, mathematical symbols, pictures, or real-life situations). When the child "visits" a particular rational-number subconstruct, he or she will be sensitive to the properties and characteristics of that subconstruct. Several of these subconstructs will be more or less intuitively accessible, and some children might be more familiar with, and think more easily "in the style of" one subconstruct (e.g., fractions), while at the same time they may be less familiar or feel uncomfortable thinking in the style of another (e.g., decimals). In general, the relations between subconstructs are poorly organized and unevenly formalized (e.g., Behr et al.1983). The whole rational-number system and its subconstructs derive some meaning from each other, and multiple representations are not just alternative means of understanding, but are viewed here as the deep structure of rational-number knowledge. Therefore, it is important to help children move easily from one subconstruct to another, to connect and differentiate between each subconstruct's characteristics and properties, and to express the same ideas using several representations such as sections of circles or rectangles, words, money, food, or time.

A major focus in previous assessments of children's development of rational-number concepts has been the role of manipulative aids including Pattern Blocks, Fraction Bars, Cuisenaire Rods, Number-Lines. The materials were used to facilitate the acquisition of rational-number concepts and representations, as the child's understanding moved from the concrete to the abstract. However, the psychological analysis done by Lesh et al. (1983), for example, showed that manipulatives were just one form of presentation in the large representational system, and that the other modes of representation (the symbolic, written, or real-life situations) also played a very important and variable role for different thinking styles, and in children's acquisition and use of these concepts. Different materials and activities were found to be useful for making models of different situations, and no single manipulative aid was found to be the "best" for all children, for all rational-number situations, or for translating fractional representations (Behr et al., 1983).

Using Logo to create and combine mathematical representations

Consistent with Lesh's psychlogical analysis of representations, the ISD project did not focus on children's working with one subconstruct or any most powerful representation (e.g., as in the Number-Lines or Pattern-Blocks curricula). Instead, ISD provided children with an environment in which they

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could work and explore relations between several representational modes (e.g., pictures and symbols), combining the different rational-number subconstructs (e.g., connecting 50% and 1/2), and translating between several representational modes (e.g., designing a screen that combined both graphical and written representations for the fraction 1/2, and the decimal 0.5).

For example, in Logo a child can program a simple picture of a circle region divided into fourths and, using different flashing colors, shade in two of these fourths and have them blink on and off in order to show a representation of two-fourths. The child can add the written words "two-fourths," which is translating pictures into written words. She can add another picture--a large round clock with an animation of the clock's big hand moving slowly from the number 12 to the number 3--and write, "this is one-fourth of an hour," then move the hand from number 3 to the number 6 and write, "this is another fourth of an hour"; or write on the screen, "one-fourth of an hour is 15 minutes, two-fourths are 30 minutes" (this is a translation of the pictorial representation of time or clocks into words, but it is also a representation of a real-life situation and its translation into fractions).

Another example, taken as these all are from the children's actual projects, is to program a picture showing a one-dollar bill with four quarters underneath. Two of these quarters be highlighted in different colors, and be animated to "walk" around the screen and "sit" beside the written words "two-fourths of one dollar." Another approach would be to compose a musical tune, then play a half of it, a fourth of it and so on. We can imagine a variety of representations, from pizzas to gears, from musical rhythms to body movements. As some of the children put it, "Fractions are everywhere." "Fractions can be put on anything!"

The ISD children used Logo to make their own representations of rational-number concepts for teaching. In so doing, they were trying to make the representations that would serve as good pedagogical aids for other children. By becoming designers of instructional software, the children gained distance and perspective in two senses. In the first place, they were dealing not with the representations themselves, but with a Logo representation of the representations. Moving between representations was subordinated to programming good examples of representations. Secondly, the children programmed, not for themselves, but for others. They had to step outside and think about the other children's reactions. This distance and perspective provided children with control of the process of learning and moving between representations, contrasts with being put through the paces of an externally-conceived sequence of learning.

In summary, ISD recast fractions learning in essentially two ways: 1) it emphasized more involvement with the deep structure (representations) over the surface structure (algorithms) of rational-number knowledge; and 2) it made fractions learning instrumental to a larger intellectual and social goal, that is, having children explain what they think and learn in an interactive lesson for younger children.

Learning more can be easier than learning less

There are certain drawbacks to implementing instructional software design activity in a school's curriculum. Software design is a time-consuming and quite a complex enterprise for a teacher to handle, and it is not yet clear how it can fit into the average class schedule. Also, it might cause problems in children's learning of other subjects in the school, and, at the present time, it is not very clear which school subjects would lend themselves best to this complex process of learning. However, the goal in the ISD Project was to experiment with one topic (fractions), and make it possible for children to learn through designing and programming instructional software much as they would do in a professional environment. The learning processes of instructional software design offered major changes in the conditions for learning:

> Knowledge about computation [such as programming] and the sciences of information [such as control over one's own processing, metacognition, and constructing of information] have a special role in changing education. Such knowledge is important in its own right. It is doubly important because it has a *reflexive* quality--it facilitates other knowledge...The reflexive quality of information science offers a solution to the apparent impossibility of adding another component to an already full school day. If **some knowledge facilitates other knowledge**, then, in a beautifully paradoxical way, more can mean less... The idea that learning more science and math means necessarily learning less of something else shows a wrong conception of the integration of these subjects into knowledge and cultures. They should be **supportive** of the other learning. It should be possible to integrate at the same time, blocks, learning of science, mathematical concepts, art, writing, and other subjects. **If two pieces of knowledge are mutually supportive it might be easier to learn both** [at the same time] **than to learn either alone** [Papert's emphases, *Constructionism: A New Opportunity for Elementary Science Education*, 1986, p. 2).

In ISD this meant that Logo and rational-number concepts were mutually supportive of each other and contributed to each other while the child was in the process of learning them. More over, young children learned fractions Logo programming, instructional designing, planning, story-boarding, reflection, self-management, etc.--all at the same time and in a synergistic fashion. ISD integrated these different kinds of knowledge and disciplines.

Learning by teaching and representing

It has been observed that the best way to learn a subject is to teach it. Many of our learning experiences as learners, teachers, and researchers, revealed to us at least one principle: that we learn most effectively from teaching or explaining something to another person. Building models and representing and explaining knowledge is essential to all forms of knowing and understanding. For example, teachers have occasionally told us that they had "finally understood something today for the first time" when a student asked for an explanation of something he did not understand. Some of our friends (professional computer programmers) at MIT have told us that they "really" learned how to program when they had to teach it to someone else--or when they were involved in a real, complex, long, and meaningful programming job. Many university professors choose to teach a course on the theory or topic of their research while they are actually working on it; through their processes of teaching and discussing their work with their students, they also identify and revise their own ideas and theories.

Fourth-grade children seldom have the same opportunity. Peer teaching can be used to take a small step in that direction. ISD took a much larger step. It provided children with similar conditions for learning as those of the MIT computer programmers, university professors, or professional software designers. These people gain expertise and learn concepts and skills by actually experiencing and exercising them in long-term, personal, professional, meaningful, and complex contexts; they acquire a deeper understanding of their knowledge and of their professions by communicating their knowledge to less experienced people; they learn about their own theories by teaching them; they learn about production by producing; and they learn about a topic by designing a videodisc or a piece of software for it. Finally, the following section from Debbie's Case (Harel, 1988, 1989) illustrates this principal:

> Consider Debbie, a black student with low mathematical scores in general, and according to the pre-test given to the experimental and two control groups (Harel, 1988). After a whole month of explaining about fractions by using different geometrical shapes divided into halves and fourths, Debbie chose to teach about an idea of a different, more "philosophical," nature than how to cut a shape into thirds or how to add a third and a half. She chose to explain that "there are fractions everywhere...you can put fractions on anything." For teaching about this idea, Debbie designed a representation of a house, a sun, and two "wooden wagons." A few weeks later, Tommy's House appeared, and then Paul's. The idea that it is important to teach others that "fractions are everywhere," and that one could "find fractions in regular human things" was spreading around the "Design Studio." Michaela and Sharifa used Debbie's software and received her full set of explanations about it. However, they chose to teach it in another way. Sharifa selected to represent fractions by using a clock, teaching her users that "Half an hour is a half of ONE hour...and this is a fraction too." And Michaela chose to teach through using a representation of "two measuring cups filled with different quantities of orenge juice, water, or flour--depends on the fraction..." Debbie's, Michaela's, and Sharifa's struggles to make sense of a fraction as an *idea*, or as a *thing*, were carefully traced. It took them a whole month to "separate" from the school's or the teacher's knowledge and to start relating to fractions in a more personal way. But

what is more relevant here is, for example, that Debbie's ability to manage fractions--including standard classroom manipulations--improved during this Project. Why?

At this stage we can come up with several related explanations. Perhaps the explanation is entierly affective: Through thinking about teaching and explaining fractions to other children, Debbie (like Sharifa or Michaela) developed a new relatioship with fractions, felt at ease with them, and so could bring herself to think about them even when given a more formal type of a problem to solve. Or perhaps, she had become more fluent in her ability to find representations of fractions, which enabled her to think about them in many, much more "tricky" situations. But whatever the explanation is, it is quite clear that Debbie's ability to work with fractions improved considerably--from a project that was entirely self-directed, and gave her no "feedback" in the form of marking responses right or wrong. In fact, this project was quite different from the kind of class work that had failed to elicit from Debbie quality thinking about fractions.

Debbie is not the only one. Before ISD had began, a battery od standardized tests in the area of fractions and a lengthy interview were conducted with all the students in her class and in two other control groups. Consistent and higher improvements were found among the experimental class on tests of fractions manipulations and algorithms--even though the experimental students had not received any more instruction on fractional manipulations and standard algorithms than the two control groups.

Debbie's experiences and their results (see Apendix) are strong examples for what we mean by constructionism, learning by teaching, and learning by representing. It allows us to contrast what Papert describes as "constructionist" and "instructionist" uses of the computer (Papert, 1986, 1989). The computer did not in any sense deliver instruction to Debbie; it did not teach her anything. Instead, it created a context in which she could learn about fractions through a lengthy process of engaged work. The fact that she was making something that she cared about gave purpose to her activity and focused her engagement.

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PROPORTIONAL REASONING STRATEGIES: RESULTS OF A TEACHING EXPERIMENT USING CONCRETE REPRESENTATIONS

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We contrast traditional approaches to the teaching of proportional reasoning based on the writing of algebraic equations with those based on a more concrete representation using computer graphics. Setting up an equation seems not to require a solid conceptualization of the situation being modeled and typically employs a syntactic solution process that is used as a black box. The latter engages a richer conceptualization and simultaneously provides a semantically transparent means for the solution process.

The multiplicative structures curriculum in the middle grades, covering multiplication, division, rates, ratios, and proportional reasoning, is widely seen as the locus of serious curricular and student difficulty. We report on of data gathered in intensive teaching experiments involving seven 6th grade classes: four experimental spanning three ability levels and three control, two at the middle ability level and one at the high ability level. We concentrate on two experimental-control pairs of classes taught by each of two highly experienced teachers at the middle ability level, and the second of three interventions involving these classes, slightly more than two weeks in duration. The school serves an upper-middle class suburb, with approximately 10% minority students bused from the inner city. Our population is younger by one to three years than those usually studied, and our curriculum was conceptually more challenging than the standard 6th grade curriculum.

These data trace the stability of erroneous additive strategies in the solution of proportion problems, and the differences in proportional reasoning that follow from differences in instruction, where control classes used the usual equation notation for proportions and the experimental computer based instruction used a variety of more concrete notations. This paper follows Kaput, et al. (1988), which reported in detail on one student's shift from consistently additive thinking to a multiplicative strategy in the context of a particular concrete model of the discrete ratios involved. While that earlier report also indicated that such shifts took place among other students who had been studied, this report provides more quantitative data on its occurrence. The experimental classes used specially designed software and materials comprising a "concrete-to-abstract software ramp" whose development and theoretical rationale have been reported on in recent PME meetings, e.g., see (Kaput, et al., 1987, 1988).

Whole-class written pre- and post-tests were given before and after each intervention as well as a general test prior to and following the entire teaching experiment. Several students from each class (both experimentals and controls) were tracked closely throughout all interventions. They were interviewed on selected tasks during the week following each written test and those in the experimental classes were observed closely, especially during computer laboratory sessions, which comprised about half the class time. Performance on the interviews closely paralleled that on the written test.

The topic coverage in the four classes varied as follows:

In the two experimental classes, 54 modeling problems, 14 "translation" mini-problems and no pure computation problems. In one control class, 19 modeling problems, 126 "translation" mini-problems and 191 pure computation problems. In the other control class, 5 modeling problems, 23 "translation" mini-problems and 68 pure computation problems.

The control classes concentrated on using equations to model proportions, including cross-multiplication. The first control class used Function Machines, another computer environment (based on a visually oriented programming language) (Richards & Feurzeig, 1989), to construct linear functions to model proportional relationships in most of the 19 modeling problems. The modeling problems in the control classes were almost all multiplicative in nature, whereas the experimental classes dealt with some genuinely additive situations for contrast purposes in a conflict teaching episode, two of which involved geometric similarity (creating two sizes of socks).

DESCRIPTIVE DATA ON GROUP PERFORMANCE

Given such variation in topic coverage, "horserace" comparisons are not as informative as more detailed examination of student performance and the thinking behind it. Figure 1 reveals

	Pre-test		**Post-test**	
	Exp	Contr	Exp	Contr
75-110%	0	0	7	2
50-74%	0	0	9	6
25-49%	1	2	7	1
0-24%	31	32	9	22

Correct Solutions (number of students)

Figure 1

substantial comparative post-test gains, in terms of numbers of students improving percentage of correct responses, for the two experimental over the two control classes at the middle ability level. It also reveals that the material was new to all students.

Of more interest is how these mid-level experimental and control classes differed in the quality of their strategies. We classified as additive those approaches based on additive or subtractive comparisons between quantities as described in the literature, e.g., (Tourniere & Pulos, 1985). Multiplicative approaches included the unit factor strategy, the usual algebraic equation strategy, and the "boxes" strategy (described below). Computational accuracy did not affect the classification. Multiplicative approaches received higher point value than the additive one, and an intermediate value was assigned for "build-up" strategies, which are an immature version of multiplicative strategies. Based on the sum of points for each proportional reasoning problem, each student's set of responses was classified as primarily additive, multiplicative, or mixed. The pre-post comparison for the experimental classes appears in Figure 2, indicating an strong move from additive to multiplicative approaches. Note that these additive-to-multiplicative changes are similar to those reported by Hart (1984, 1988), although not quite as complete probably due to the younger age of our students.

Pre-Test			
Mult	0	0	0
Mix	1	5	9
Add	5	3	8
	Add	Mix	Mult

Post-Test

1 Block gain: 32%
2 Block gain: 32%

Significant gain total: 64%

Figure 2

About 35% of the students in the two control classes showed gains, with most of these (75%) occurring, interestingly, in the Function Machines, class.

CONTRASTING EQUATIONS WITH MORE CONCRETE APPROACHES

More detailed data relate the different patterns of responses to the different types of tasks. We hypothesized that traditional instruction based on writing an equation for the proportion would yield correct surface performance on routine missing value proportion problems (e.g., through solving equation representions of proportions), but would leave underlying conceptions relatively unchanged, as reflected in the return to primitive strategies in more difficult problems, non-routine problems, and especially problems that involve both additive and multiplicative structure in combination or are otherwise not amenable to formulaic approaches. The data confirm these hypotheses in two ways, first in the tendency of control students to revert to additive approaches in more difficult problems and, second, in the tendency to apply the equation writing strategy inappropriately on variants of missing values problems.

We now review the two approaches to solving missing-value proportion problems with respect, first, to setting up the representation and, second, to employing the representation to solve the problem. We shall identify differences in the relationship between the conceptualization of the situation being modeled and the different conceptualizations of the two model themselves.

THE EQUATION WRITING APPROACH

The standard solution strategy is to set up an algebraic equality of two ratios in fraction form: A/B = C/D, a comparison of multiplicative comparisons. One ratio is provided or to be inferred directly from given information, while the other is incomplete, indicated by the use of a symbol for an "unknown" in one of the four places in the equation. Consider the following "toys" problem.

A toy store sells 5 matchbox cars for every 8 stuffed animals. How many stuffed animals are sold if the store sells 40 matchbox cars?

To set up an algebraic equation, the student must first identify two ratios to be compared. As is well known (Karplus, et al., 1983; Vergnaud, 1983), there are two types of (within ratio) comparisons possible. On one hand there is the "within measure" or "scalar" comparison, where one compares like quantities, in this case, cars to cars and animals to animals. Each of these can be written as ratios in two ways, as 5 cars/40 cars and 8 animals to X animals, or their inverts. On the other hand, one can compare "across measures" or in a "functional" comparison, where one compares cars to animals or animals to cars. Written as ratios, these look like 5 cars/8 animals and 40 cars/X animals or, respectively, their inverts.

In this case, the identification is relatively explicit in the problem statement - we are comparing numbers of matchbox cars sold with the number of stuffed animals sold. (In other problems, especially in geometric similarity problems, choosing an order for the comparison does not at all follow so easily from the problem statement.) Meticulous teaching of this strategy usually calls for the student to include units for the numbers, so the ratio is to be expressed as an intensive quantity: 5 cars/8 animals. The purpose of the units is to make more explicit the direction of the (within-ratio) comparison, and, indeed, the fact that a comparison is actually being made.

We are now ready for the second level of the comparison, the comparison between ratios to be embodied in the equal sign. One must now write the second ratio, which reduces to determining the position of the unknown in the fraction. It is here where the units already written can play their role, by helping the student determine the directionality of the comparison - which is assumed to be the same across both sides of the equation. Again, meticulous teaching makes this point explicitly. In the toys problem, we are comparing cars to animals and are told that the number of cars is 40 (although the word order in this part of the problem statement is reversed), so we write the "40 cars" in the numerator of the fraction matching the left side, and write "X animals" in the denominator, where the units likewise match the left side denominator.

At this point, having written the equation representation, the usual syntactically based transformations of this representation comprising the solution process (cross multiply and divide) constitute a black box cognitively independent of the conceptualizations that led to the initial equation. In particular, students have little understanding of the referential content of the intermediate steps, of what 5*X = 8*40 might stand for, even if the units ("car-animals") are included. However, if one uses more semantically guided transformations based on maintaining equality between equivalent ratios as one adjusts one or the other side, then the process amounts to an elaboration of those conceptualizations. Controls were taught the syntactic method.

THE BOXES APPROACH

The approach described here is based on a conceptualization of the given quantities and relations among them as groups or clusters of 5 cars and 8 animals paired in some way. This appears to be a more natural conception for discrete quantities in that it appears spontaneously among students for whom it has not been taught, and it appears to coexist with an equations approach after the latter has been taught. This contrasts with the equations approach, which appeared only when taught in our data as well as that of Hart (1988). We chose to capitalize on students' tendency to think of a proportion situation in the paired-groups style by providing a notation system that is consistent with it and exploits its transparency as the basis for reasoning with it - see Figure 3.

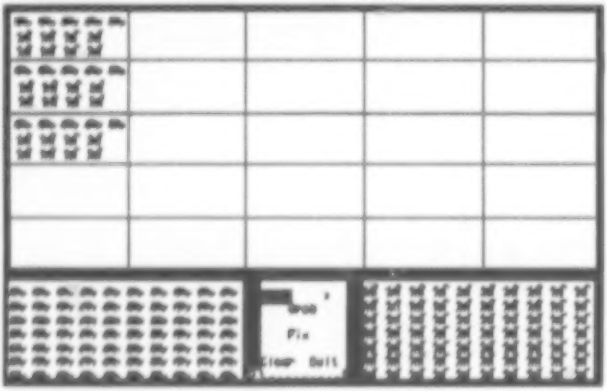

Figure 3

Experimental students were provided a computer environment that enabled them to solve problems such as the toys problem by dragging sets of 5 cars and 8 animals into individual cells as indicated until the specified number of cars (40 in this case) has been matched with an appropriate number of animals. By counting corresponding animals (often by multiples of 8), the student is then able to answer the question. This strategy is can be done in both scalar and functional styles, e.g., by dragging cars and animals either separately or together, respectively. Another part of the computer environment supports incrementing the values of the car and animal quantities simply by clicking on a MORE button, with the results shown not only in the rectangular array, but in a table of data, and, if needed, in a cars-by-animals coordinate graph.

This incrementing approach can then be abbreviated and made more efficient by moving from actions on sets of visual icons representing the quantities in the situation to actions on more formal representations of the quantities. In particular, we ask:

How many "boxes" of cars and animals are needed if there are 40 cars?
The question is answered with a quotitive division:
40 cars/(5 cars/box) = 8 boxes.
The next question is:
How many animals are needed for 8 boxes?
The answer is given by a product:
(8 boxes)*(8 animals/box) = 64 animals.

Almost every student understood this strategy and employed it reliably on missing values problems involving discrete quantities. Its learnability and its spontaneous appearance are only one advantage over the more formal algebraic equations strategy. We closely examined four students who relied on the equation strategy, two each from each of the two mid-level ability control classes who were among the sample of students interviewed.

A critical difference between strategies is the relation between the initial conceptualization of the situation being modeled and the ensuing solution process. It is possible to set up an equation using a relatively impoverished conceptualization. One needs merely to distinguish the two quantities from one another and use the units in a mnemonic manner to mark the relative positions of those quantities in order that the comparison be consistent across the equality sign. One can then employ the black-box solution process in a mechanical way without further engagement with the conceptualization that yields the equation. Such an opaque process, of course, is frought with the danger of unrecognized error or inadequately interpreted computational results.

On the other hand, the boxes strategy builds directly on a quantitative conceptualization of the situation being modeled throughout, from initial set-up to final answer. A consistent difference has been observed in the interviews in previous teaching experiments when students have been asked why they believe in their answer. The equations-based solvers tend to say either that they don't know or "it came out that way," whereas boxes-based solvers tend to say, at worst, "because I worked it out." At best, they give clear explications in terms of the quantities involved. They tend to identify the justification with the method, because its rationale is transparent to them.

Another difference between the strategies appears in the brittleness of the equations strategy in the face of variants of missing values problems, such as totals problems: *In the senior class, there are 3 girls for every 5 boys. If there are 240 students, how many boys and girls are there?* Or problems built on part-whole rather than part-part descriptions: *Three out of every 5 students are wearing sneakers. If 105 students are wearing sneakers, how many are not wearing sneakers?* (In each of these, students treated them as part-part comparisons.) Or problems involving multiple proportions: *In a bowl of candy there are 3 M&M's for every 4 jelly beans and 5 gumballs. If there are 60 jelly beans, how many M&M's and gumballs are there?* (Here they reverted to additive strategies.)

For such problems, students who depended on the equations strategy tended to mis-apply it here, whereas the boxes-strategy students did so less frequently. In fact, equation solvers also tended to mis-apply the method on true additive problems. Furthermore, among four consistent equations users (who were almost 100% correct on straightforward missing value problems, the only correct solutions used a boxes style approach.

CONCLUSION

The formally-based equations approach to solving multiplicative structures modeling problems, while computationally efficient and general, suffers from the shortcomings of all conceptually opaque procedures. In particular, its success is especially deceptive to teachers because it is easy to learn and apply in a superficial way on a semantically narrow, but computationally general, set of proportional reasoning problems. Embedding the approach in strategies which distinguish variants of proportional reasoning problems, and using a more transparent equation solving process based on equivalent ratios, are likely to improve the quality of the learning outcomes.

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THE COMPUTER AS AN AID TO FORMALIZING ARITHMETIC GENERALIZATIONS

by

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Fifth-grade children, working on MacIntosh SE computers utilizing HyperCard, made progress toward abstracting and formalizing generalizations of arithmetic into algebraic statements. The students formed arithmetic generalizations from experiences solving problems with physical materials; via HyperCard's scripting language, HyperTalk, the students communicated their generalizations to the computer in algebraic form. The experience aided the students to develop concepts basic to variable, equivalence of equations, equation evaluation, and equation solving.

A large part of the difficulty children have with mathematics is due to a lack of understanding of the relationship of variables to mathematical concepts (Rothman, 1988). The creation of "text fields" and "containers" for "holding" numbers in HyperTalk addresses this problem. The twenty-week project described below was aimed at determining if the computer, given a sufficiently easy language, could be used as a means of making the variable-mathematiaal relationship clear to elementary school students. The project did encourage the connection of variable to mathematical concept and also helped with the interpretation and analysis of mathematical expressions, but seemed to have a neutral impact upon computational efficiency.

The discussion of the project will proceed with a background description of the children involved, a description of their introduction to the computer, the instructional parameters, observations, and concluding comments.

STUDENT BACKGROUND

The children involved in the project attended a private parochial school which draws it's students from the urban and suburban environs of Salt Lake City, Utah. The students represented middle to upper-middle class social elements with a smattering of minority and ethnic subgroups. The mathematics capabilities of the students as measured by Education Review

Board(ERB) examinations ranged from the 28th to the 99th percentile with an average class position at the 75th percentile in mathematical concepts. The ERB placed the children's computational abilities on the average at the 63rd percentile with a range from the 11th to the 99th percentile.

The children were instructed in arithmetic at their school, then transported to the University of Utah for an hourly Macintosh experience once a week.

INTRODUCTION TO THE COMPUTER

The student's first two experiences with HyperCard centered on the process of accessing and creating "card stacks" and exploring the "tool pallette". The third and fourth hourly sessions were devoted to creating and copying "fields", "buttons", setting the font and style parameters for field input, and "scripting" (programming) buttons. By the fifth session, the children began to "script" buttons to compute surface areas and volumes for some rectangular configurations and progressed from there to other generalizations derived from their experiences with physical models.

THE INSTRUCTIONAL SEQUENCE

The instructional program consisted of a concrete phase, a representational phase and a mental imagery phase followed by an experience in scripting the computer to accomplish specific cases of the children's generalizations (Figure 1).

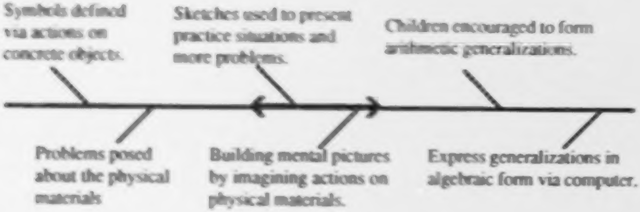

Symbols defined via actions on concrete objects.
Sketches used to present practice situations and more problems.
Children encouraged to form arithmetic generalizations.
Problems posed about the physical materials
Building mental pictures by imagining actions on physical materials.
Express generalizations in algebraic form via computer.

Figure 1.

Instruction began by asking the students to imagine a unit cube as a rubber stamp and the back of the instructor's hand as a stamp pad. The students were asked to decide how many stamps with the unit cube would be necessary to completely cover "5-rods" arranged in offset sequences as shown in figure 2.

The students worked on the sequences until they found a way to determine the number of stamps (surface area) required for any number of rods. Below are two "children's" solutions for the sequence in figure 2:

I. The first rod requires 22 "stamps". Each additional rod adds 14 stamps. When the rods are "glued" together eight stamps are lost, four for each rod. The total number of stamps needed for 20 rods is twenty-two, plus fourteen times nineteen. The number of stamps for any size stack is 22, plus fourteen times the total number of rods less one.

II. The first rod (rod 1) and the last rod (rod n) have eighteen stamps each. Each rod inserted in between has fourteen stamps. So the number of stamps in a 20-rod stack is two times eighteen, plus eighteen times fourteen. The total number of stamps for any size stack of rods is thirty six plus fourteen times the number of rods less two.

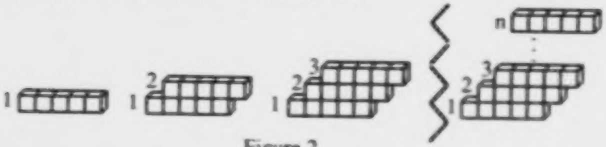
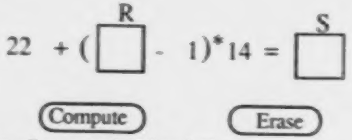


Figure 2.

The children instructed the computer to handle specific cases of generalizations they had formed by setting up two "text fields" in HyperCard. They named one field "R" in which they entered the total number of rods. The second field they named "S" to which the computer would "return" the number of stamps required to cover a given stack of rods. The children created two buttons, one to "compute" and the other to "erase". The "erase" button was "scripted" to clear the fields between calculations while the "compute" button was scripted to calculate the number of stamps required for any given number of rods. Below is an example of a student effort for the sequence in figure 2:



The "compute button" was scripted as follows:

```
on mouseUp
  put card field "R" into r
  put 22 + (r-1)*14 into card field "S"
end mouseUp
```

EQUATION SOLVING

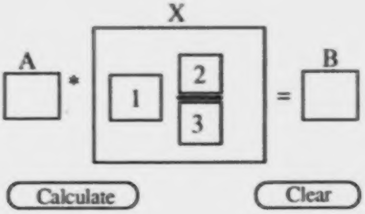
The children were introduced to simple equation solving by extending their experiences with arithmetic into "missing number" statements from which the children generalized methods of solution (Peck, 1988). Figure 3 illustrates the kinds of examples used.

a. $4 \times 5\frac{1}{2} = \square$
$3 \times 7\frac{2}{3} = \square$
$2\frac{2}{3} \times \square = 7$

b. $3 \times \square = 25$
$\square \times 5 = 21$
$75 = 9 \times \square$

Figure 3.

The children initially solved problems like those in figure 3 by using a pegboard model (Jencks, 1985). When the students were comfortable with solving equations from the model, the instructor asked the children to try to answer before constructing the problem on a pegboard or graph paper. When the children could answer from a "mental image," they were asked to instruct the computer to solve the problem for them. The students knew, for the generalization ax = b, they had to divide b by a to get the whole number part of the answer. The remainder represented the numerator and the divisor represented the denominator of the fractional portion . They had no experience with common fraction-decimal relationships so the instructor introduced them to the HyperCard instruction "b div a" and "b mod a" as way to get the whole number and remainder. Given this information, the children made five fields and scripted a "compute" button to do the computations. The following is a student example:



The "calculate" button was scripted as follows:

```
on mouseUp
  put card field "A" into a
  put card field "B" into b
  put b div a into card field "X1"
  put b mod a into card field "X2"
  put a into card field "X3"
end mouseUp
```

The large majority of students were also able to program the following generalizations:

Volume = $l*w*h$

Surface Area = $(l*w + w*h + l*h)*2$

$a/b \pm c/d = (a*d \pm c*d)/(b*d)$

$(x+1/2)(x+1/2) = x(x+1) + 1/4$

$(x+1/2)(x+1/2) = x*x + x + 1/4$

$(x + 1/2)(x+1\ 1/2) = (x+1)(x+1)-1/4$

OBSERVATIONS

As a final experience, the children were presented with missing number problems in the form ax/b = c, where a, b, and c were whole number constants as shown below:

$\frac{4*\square}{5} = 11$ | $\frac{3*\square}{7} = 5$ | $\frac{11*\square}{5} = 5$

The children were asked to work out a way to solve such equations and then write a program to instruct the computer to do specific examples. The children were separated into four groups (I - IV) based upon their conceptual understanding and management of the computer as revealed by their work, observations and personal interviews.

Seventeen children fit into group I. These children possessed a clear understanding of the role of variables as "containers" into which numbers or results could be placed. These children could explain how the computations performed by the machine related to the derived concepts. These children could script the machine and correct their errors without consulting the instructor.

Nineteen children fit into group II. These children possessed a clear understanding of the role of variables also. They had minor difficulties scripting the computer. Most of their problems related to misspelled words or omissions in scripting statements. In general, however, the children differed little from those in group I.

Four students comprised group III. These children had conceptual difficulty. They could not describe how operations with variables related to the arithmetic and the physical models used to develop it. They were dependent upon their peers and exhibited counterproductive perceptions of mathematics closely akin to those described by Frank (1988). That is, these children seemed to be victims of a rule-orientation that inhibited their ability to make logical connections between the symbols and underlying referents.

The remaining four students were placed in group IV. These students profited little from the mathematics experience, nor did they manage to learn how to deal with the computer. One of the children had an IQ of 81 and language and reading difficulties. Another of the children in group IV had difficulty interpreting auditory messages. His IQ was also in the lower eighties. He read on a second grade level. The remaining two students had flashes of insight , but seemed unable to profit from their experiences.

The thirty-six children in groups I and II were not intimidated by variables nor did they view formulas as merely mnemonic devices for remembering computational procedures as is often the case with students in beginning algebra, or even some university students the author has known.

Five children in group II exhibited a peculiar behavior. The error patterns they had acquired from pre-project experience persisted. Even though the children could conceptually describe the ideas, explain what was going on, and program the computer correctly, the error patterns returned on each succeeding quiz. Holt (1982) and Davis and Mcknight (1980) described children who were familiar with physical models for mathematical concepts, but failed to use them as a basis for decision making. The quiz responses of these five students seem somewhat reflected the conclusions of these studies. The experiences of these five students with physical materials and the Macintosh failed to help them overcome their previously learned error patterns, at least within the time frame of the project.

CONCLUDING COMMENTS

The project did not represent a strict test of the potential of the Macintosh as an aid to formalizing. Adequate controls and comparison groups need to be established and the range of variables limited to qualify as a scientific effort to determine exact effects. A more careful study is planned with the needed computers housed in the school where they will be readily available to support the instructional program. Nevertheless, the project results suggest that thirty-six of the fifth grade students (groups I & II) developed a useful concept of variable, a rudimentary notion of equivalent equations, equation evaluation, and equation solving from their computer experiences and they were able to relate these notions to the arithmetic they had generalized from their experiences.

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NUMBER CONCEPTS

THE KINDERGARTNERS' PROCEDURAL UNDERSTANDING OF NUMBER: AN INTERNATIONAL STUDY

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This paper reports some results of an international study (Montreal, Paris, and Cambridge,Mass.) on the kindergartners' procedural understanding of number. Specific questions and tasks assessed the range of their enumeration skills, their counting-on skills and their utilization in both cardinal and ordinal contexts, as well as their understanding of the counting backwards procedure.

Our investigation of the kindergartners' numerical knowledge is now in its fifth year and our results reflect new approaches both at the theoretical level and at the methodological level. At the theoretical level, our research has started with an epistemological analysis of the number concept (Herscovics & Bergeron, 1988). This provided an overview enabling us to perceive number as a conceptual scheme, that is as a network of related knowledge together with the "problem-situations" in which it can be used. Regarding our methodology, we have adopted the clinical approach used in case studies but have tried to go beyond a few individual cases and have used larger samples averaging thirty odd children in order to identify likely patterns of thinking.

Using the above analysis we have developed a sequence of about forty tasks aimed at uncovering the child's numerical knowledge.The samples used in our study involved 29 Parisian kindergartners of average age 5:8 whose school was situated in a lower socio-economic neighborhood (lower middle class and working class); 30 kindergartners of average age 5:10 whose school was located in a lower socio-economic neighborhood in Cambridge, Mass.; 14 of these children were in regular classes whereas 16 of them were following an activity oriented program for early childhood based on Mary Baratta-Lorton's **Mathematics Their Way** (1976); 32 kindergartners of average age 6:2 from 4 different schools in the Montreal area, two being situated in higher socio-economic suburbs and two located in lower socio-economic neighborhoods. For the overall project, which dealt with all the different aspects of understanding number, three to four individual interviews lasting on average 30 minutes were carried out with average children selected by the

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school authorities. The interviews took place in Paris between the end of February and the beginning of April 1988, and in Montreal and Cambridge between the end of April and the beginning of June.

UNDERSTANDING OF ENUMERATION

Pre-requisite to any mastery of the enumeration procedures is the child's memorization of the number word sequence. However, prior research has shown that a majority of kindergartners perform better on the enumeration of a large set of objects than on the mere recitation of the number-word sequence (Anne Bergeron et al.1986). Thus in order to assess the extent of their knowledge of the number-word sequence, a set of 76 chips was provided with instructions to "Count as far as you can". The following table indicates the distribution of their enumeration skills.

	N	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70+	Average
Cambridge										
Regular classes	14	0	0	3	3	0	1	3	4	53.7
Lorton classes	16	0	0	0	0	0	1	1	12	70.8
Total	**30**	**0**	**0**	**3**	**3**	**2**	**2**	**4**	**16**	**62.8**
Paris	**29**	**1**	**10**	**6**	**5**	**1**	**3**	**0**	**3**	**32.4**
Montreal										
Higher Soc.Ec.	16	0	4	1	4	1	0	4	2	45.1
Lower Soc.Ec.	16	1	3	4	4	0	1	1	2	37.3
Total	**32**	**1**	**7**	**5**	**8**	**1**	**1**	**5**	**4**	**41.2**

What is striking at a first glance is the similarity between the Parisian and Montreal samples, this, in spite of the fact that the French children were six months younger than the Canadian ones . But even more striking is the shift in the distribution of the Cambridge children. Not a single child is in the 0–19 range, in contrast with the 25% and 37% in the other two cities. Moreover, half the Cambridge children can count beyond 70, compared to 12.5% and 10.3% in the other two cities.

The distributions provide another interesting fact. It seems that for the Montreal and Parisian children, as well as for the Cambridge regular classes, the number 39 constitutes a temporary plateau: 65.6%, 75.8% and 42.9% respectively are within the 0–39 range. Perhaps this might indicate that these children have not yet learned the sequence of multiples of ten. That two decades, from 20 to 29, and 30 to 39, are sufficient for the generalization of the decade structure, seems evident from the fact that when the children learn their multiples of ten, their range jumps up to the sixties and seventies. Few of them remain in the 40 to 59 range.

A greater frequency of the Parisian children in the 50-59 range might be explained by a lack of knowledge of the multiples of ten beyond 50. One might conjecture that the 5 Montreal children in the 60-69 range (16.7%) have difficulties with 70 since in French, the tens pattern changes (..., cinquante, soixante, soixante-dix, ...). However, the data does not bear this out, since in the regular Cambridge classes, 3 out of 14 children (21.4%) are in the same range.

UNDERSTANDING COUNTING-ON

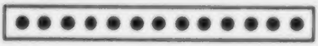
Fuson & Hall (1982) report that when the number word sequence becomes a breakable chain, children can start **counting-up** (reciting-up) from a given number and that this skill translates into a cardinal operation, that of **counting-on** in the context of addition (p.52). In our study, we have experimented numerical tasks requiring counting-on in non-additive situations involving both cardinal and ordinal contexts.

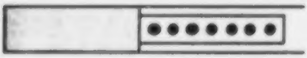
A global look at the results indicates that nearly all kindergartners can recite up from a given number (only 7 out of 91 cannot) and that most of them do not even need a running start. Comparing the performance in the three cities shows that nearly all (90%) the Cambridge children can start at 12, that about two thirds of the Montreal children (68.8%) , and about half of the Parisian sample (48.3%) can also do so. These differences can easily be explained by the emphasis on counting found in the Cambridge school and by the age difference of the Parisian children who were six months younger than the Montreal ones.

Having assessed the children's reciting-up skills, some special tasks were designed to determine their spontaneous use in the solution of cardinal and ordinal problems. Initially, these tasks were similar to the one used by Steffe, von Glasersfeld, Richards and Cobb (1983). Each child was presented with a row of 11 chips glued to a cardboard, the interviewer stating:

Here is a cardboard with some chips. Look, I'm putting it in this bag

(while inserting it in a partially opaque plastic bag)





Look, six chips are hidden here (indicating the opaque part)

Can you tell me how many chips are in the whole bag?

The results indicate that with the exception of the Lorton classes, the predominant procedure used was that of figural counting: 50% of the children used it. It should be noted that for the regular Cambridge classes, the procedure was used successfully at a rate of about 40%, but with all other groups, the success rate was about 70%. Counting-on was the main procedure for the Lorton classes (56%). And in all cases, including the other samples, its use guaranteed success. In spite of the care taken in formulating the questions, nearly a third of the Parisian sample focused only on the visible chips.

Following the cardinal task, the same material was used on an ordinal task. It required locating the chip corresponding to a given rank. Using the same material as before, the interviewer asked:

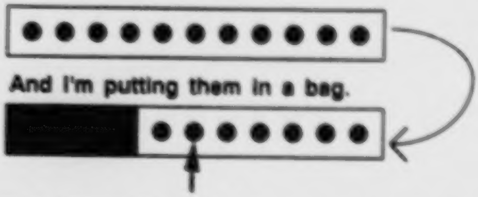
Remember, there are six chips that you can't see. Here is the first one
(pointing out the one on the extreme left of the hidden part)
Can you put this little arrow next to the ninth chip?

The data show that once again, with the exception of the Lorton classes, figural counting is the most common procedure, its use ranging from 65% to 75%, as compared with 43% to 56% on the cardinal task. The Lorton group again favored counting-on. The ordinal task seems to have been better understood by the Parisian children since only two of them restricted their counting to the visible chips; six of them used counting-on, as opposed to only one for the cardinal task.

Although most children can recite-up, the use of the counting-on procedure is relatively low, except for the Lorton classes, even if we use the best performance on ordinal tasks. The number of these children is 4 out of 13 (30.8%) for the regular Cambridge sample, 7 out of 23 (30.4%) for the Parisian children, 5 out of 16 (31.3%) for the Montreal sample from higher socio-economic neighborhoods, and 6 out of 16 (37.5%) for the other Montreal sample, as compared with 12 out of 16 (75%) for the Lorton group. Quite clearly, only about a third of the children who possess the reciting-up skill think of using it in the above tasks.

These results bring into question the meaning of counting-on for most of these children. To investigate their interpretation, a simple task in which they were asked to count-on was proposed. The interviewer presented them with 12 chips glued to a cardboard. This cardboard was then inserted in a partially opaque plastic bag so that 4 chips would no longer be visible:

Here is a cardboard with chips glued to it



Look, there are some hidden chips. When I counted them, I started from here (pointing to the first hidden chip on the left) **and when I got here** (putting a small arrow next to the sixth chip) **this was the sixth. Can you continue counting from here on, from the sixth one?**
When the counting was completed:
Can you tell me how many chips are in the whole bag?

The results show that out of 87 subjects who could count-on, only 33 of them (37.9%) could tell how many. Thus a full 62.1% could not ! Of course, this brings into question the children's interpretation of the counting-on procedure. The surprisingly poor performance on this task might be explained in terms of two conjectures: (1) Perhaps it is the non-visibility of some of the objects that affects the children's capacity to relate the counting-on procedure with the cardinality of the set; (2) Perhaps it is their need to still establish a one-to-one correspondence between the number-words and the objects.

A closer look at the performance of the Lorton classes brings about some further questions. Out of 16 children, 9 used the counting-on procedure to solve the cardinality question. Of these 16 subjects, 8 could tell "how many?" after counting-on. At a first glance, this would conform to our expectations. But a more detailed analysis shows that of the 9 subjects who counted-on in the first cardinal task, 6 of them did not succeed in the second cardinal task: although they counted-on from 6 to 12, they could not state the cardinality of the row. One could attribute this lack of consistency to a certain instability of these children's interpretation of the counting-on procedure. However, an alternate hypothesis might be suggested on the basis of the difference between the two tasks at hand. In the first one, the child starts counting-up from a cardinal number ("There are six hidden chips") whereas in the second one, the start is from an ordinal number ("When I counted them... this was the sixth one"). This third conjecture points to a possible gap in the children's integration of the cardinal and ordinal aspects of number.

UNDERSTANDING COUNTING BACKWARDS

In the Fuson & Hall (1982) hierarchy the next two skills are counting backwards from a given number down to 1, and counting backwards from a given number and stopping at another given number. These procedures prove to be important for, as mentioned by Carpenter & Moser (1984), they are often used by children up to Grade 3 in the solution of subtraction problems. Our investigation follows the same sequence as in the last section: the recitation skills, the spontaneous use of the corresponding counting procedures in the solution of numerical problems, and finally, the mastery of the counting backwards procedure itself.

In the assessment of the backward recitation, children were asked:

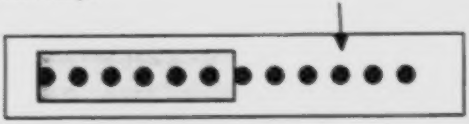
Can you count backwards starting from 12?

We did not insist on starting at 12 since most children find it easier to count back from 10. Even then, if they found it too difficult, we suggested trying from 6 or 7. One finds that reciting backwards down to 1 is mastered by all the kindergartners in the Cambridge samples, that 18 out of 29 of the Parisian children (62.1%) and nearly all the Montreal subjects (30/32 = 93.8%) have managed to do so. However, the distribution indicates marked degrees of achievement as shown by the ability to recite backwards from 12 (23, 7, and 12 respectively)

To determine if children who could recite backwards the number-word sequence would spontaneously use this skill in numerical situations, the following problems were set:

The interviewer presented a row of 12 chips glued onto a cardboard and then, in front of the child, proceeded to hide 6 chips on the extreme left with another small cardboard strip:

Look. I've hidden some chips



(a) When I counted them, I started counting from here (indicating the extreme left of the hiding carboard), **and when I got here** (placing the arrow next to the 10th chip), **this was the tenth, the number ten chip.**

Do you have a way to find out how many are hidden?
(If the child could only count back from 6, two chips were hidden and the arrow was placed next to the sixth chip)

(b) Hiding 3 chips on the left and replacing the arrow on the 10th chip
Now I've hidden some chips again. Look, I'm putting back the arrow next to the tenth chip. Can you show me where is the seventh chip?
(If the child could only count back from 6, 2 chips were hidden, and the arrow placed next to the 6th chip while asking to be shown the 3rd one)

The data show that the success rate on the cardinal task was very low. Among the 78 children who could recite backwards to 1, only 12 (15.4%) thought of counting backwards to find the hidden part. The ordinal task was handled with much greater success. About half the children who had the pre-requisite reciting skills did use them in the solution of the ordinal problem. Theoretically, the ordinal task should be somewhat more difficult, since in finding the 7th chip, the child must know when to stop. The surprisingly different results on the cardinal task again raise some interesting questions. Could they be attributed to a lack of integration of cardinality and ordinality? When the 10th chip is shown, the child needs to count back from an ordinal number and must shift to a cardinal frame when finding the number of hidden chips.

CONCLUSION

Regarding the basic enumeration skills, counting from 1, the Cambridge samples proved to be ahead of the Montreal ones who themselves were slightly ahead of the Parisian one. The more sophisticated procedure of counting-on was mastered by most of the subjects. However, few of them used it in the cardinal and ordinal tasks, with the exception of the Lorton group. The particularly surprising inability, by 62% of the pupils, of answering the question "how many?" after they had counted-on as requested, points to a lack of integration of the counting-on procedure with their understanding of cardinality. Another problem of integration was evidenced by the children's failure to find the cardinality of a hidden part by counting backwards from an indicated rank. While these are interesting problems at the research level, they should not obscure the fact that in all the samples, these kindergartners possessed a broad and varied knowledge about number. For instance, many could double count forwards and backwards.

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EFFECTS OF INSTRUCTION ON NUMBER MAGNITUDE

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Fourth-grade teachers taught specially prepared units on decimal numbers and fractions that emphasized comparing these two types of numbers. A written pretest and individual interviews indicated that students had little understanding of number size concepts. Interviews held immediately after instruction and again at the end of the school year showed that students could compare fractions and decimal numbers and could give reasonable explanations for their answers. A curriculum that spends more time developing these concepts might help children acquire better number sense.

Concepts related to number magnitude have received little attention from researchers. Investigators who have looked at number magnitude have usually had a broader research focus for their work. Consequently, their research reports give only a limited amount of information about children's number size concepts. Also, their research is frequently limited to one type of number, usually at one grade level. Even so, their results provide valuable insights into children's understanding of numbers and related symbol systems.

Studies that included comparing and ordering decimal numbers clearly show how students overgeneralize features of whole numbers. In the Sackur-Grisvard & Leonard (1985) study, fourth graders judged 3.7 to be smaller than 3.53 because 37 is smaller than 353 or because 7 is smaller than 53. Almost half of the sixth and seventh grades tested by Hiebert and Wearne (1986) selected .1814 as the largest number of the set .09, .353, .3, and .1814. Young children's understanding of order and equivalence of fractions is also dominated by whole number knowledge of ordering (Behr, Wachsmuth, Post, & Lesh, 1984; Kerslake, 1986). Children focus on either the numerators or the denominators, and order the fractions accordingly. Thus, 3/5 would be considered larger than 3/4, because 5 is larger than 4. Sowder and Wheeler (1986), in a study that considered performance on number magnitude tasks at grades 4, 6, 8, and 10, found that overgeneralization diminishes as children proceed through the grades, but that the ability to compare fractions is much slower to develop than the ability to compare decimal numbers. This fact is not surprising, since comparing fractions cannot be as easily stated in terms of rules to follow.

Curriculum developers have also paid scant attention to the potential power of work on number magnitude for increasing understanding of numbers and their systems of symbolization. Textbooks contain few lessons on comparing and ordering fractions or decimals, and when they do occur, they frequently do not focus on number meaning. For example, a widely used fourth grade mathematics text with a 1987 copyright has one lesson on comparing decimals and one on comparing fractions. The decimal number lesson compares numbers with the same number of decimal places, such as 3.6 with 3.8, or 4.83 with 4.92. The fraction lesson compares fractions with the same denominator, gives rules and problems comparing a fraction with 1, and finally has students compare pairs of fractions by first locating them on a number line that appears in the text with all the fractions located and named. None of these exercises required any real understanding of fraction and decimal size.

The purpose of the study described here was to investigate the effect of instruction on comparing fraction and decimal numbers. In particular, we wanted to know whether instruction that focused on the meaning of the numbers being compared would lead to increased understanding of fraction and decimal numbers and associated symbols.

Subjects and Procedure

Students from three fourth grade classrooms participated in the study. Two classes were from a school located in a middle-class neighborhood, the other from a school located in a slightly lower socioeconomic community. The schools were selected on the basis of willingness of principals and teachers to participate. One teacher from each school taughts units on decimal numbers and on fractions. Both units were prepared by the investigators. Students from all three classrooms were given a written pretest. In each of the two classrooms where teachers used our instructional units, ten student of average ability, as ascertained by the teachers, were selected for more intensive study. All 20 students were interviewed at the beginning of the study in January, 14 were interviewed after the decimal unit, 15 after the fraction unit, and all 20 were again interviewed at the end of the study in June.

Instructional Units

The decimal unit contained seven lessons, and extended over approximately ten 50-minute periods. The first five lessons introduced students to base ten blocks by requiring them to represent whole numbers with the blocks, and finally to represent decimal numbers with the blocks. The fifth lesson also included representing and comparing numbers such as 0.4 with 0.40; and 0.4 with 0.04. The sixth lesson required students to represent money as decimal numbers, and to compare amounts. The final lesson (requiring about two class periods) focused on comparing decimal numbers both by first representing them with blocks, and then comparing decimal numbers without the blocks. Each lesson began with a long segment developing the concepts in the lesson and presenting examples. Students were required to complete a set of problems in class. The problems were then displayed on an overhead projector and discussed by the class. A second set of problems was usually given for homework and discussed the following day.

The unit on fractions contained nine lessons, and the teachers completed the lessons in approximately ten days. Fractions were introduced as quantities, so that a fraction symbol was shown to represent some amount, just as a whole number symbol did. In the second lesson, students were given clock faces and asked to draw in a half hour, a quarter of an hour, and a third of an hour. Later, using pieces of circles and squares, fractions with the same numerators and different denominators were compared, then fractions with different numerators and same denominators were compared. One lesson focused on equivalent fractions. Following that, students were asked to compare fractions with one-half, and formulate rules for when a fraction is greater than one-half and less than one-half. Pairs of fractions with the same denominator were presented, and students were asked to chose the fraction closer to one-half, or closer to zero, or closer to one. Fractions were located on a number line, then compared. In the eighth lesson, pairs of fractions were compared by choosing the most appropriate technique. Finally, fractions and decimals were compared in size. The format of the lessons was like that for the decimal unit.

Student Performance Before Instruction

Only 9% of the 86 fourth-grade students who took the written pretest identified 3.7 as the larger of 3.7 and 3.53. This is the same problem used in the Sackur-Grisvard and Leonard (1985) study mentioned earlier, and our results were comparable to the results found there. On fraction items, given 1/3 and 1/4, 5% selected 1/4 as the number closer to 0; 7% selected 2/3 as the larger of 2/3 and 2/4;

but 63% could correctly identify 4/3 as larger than 3/4.

The interviews included tasks with whole numbers, decimal numbers and fraction numbers, and focused on place value, symbol meaning, number line concepts, and number magnitude. Some mental computation tasks were also given. Only the tasks directly pertaining to number magnitude are presented here.

On the initial interview, a decimal item asked whether one meter was closer to 0.9 meter or to 2 meters. Most students selected 0.9 meter, but this result is suspect since the item was read aloud to students as nine-tenths. Students who did answer this question correctly were unable to select the correct points on the number line corresponding to 0.3 or to 0.8, where the letter A was placed at 0.3, B at 0.8, C at 1.2, and D at 3. Of particular interest here is that one class had completed a unit at the beginning of the year on decimal numbers, while the other class had not yet received any instruction on decimal numbers, yet there was no real difference in student responses between the two groups.

In a fraction item, students were told that two children were painting a fence, one on either side. The students were asked which of the two children was ahead after one had painted 1/5 of a side, the other 1/8 of a side. Two of the twenty students could answer correctly but only one of the two could give an explanation indicating some understanding of the relative size of the two numbers. Ten students were then told that after two more hours, one had painted 1/2 of a side, and the other 2/3 of a side. Three of these ten students answered correctly and justified their solution by shading in the rectangles given as the two sides of the fence. The other ten students were to similarly compare 1/2 with 3/10. Only one student gave a correct answer with a reasonable explanation: "It's half, and he only painted 3 out of 10."

Student Performance After Instruction

Immediately after instruction, 14 students were asked which was larger, 14.7 or 14.26. Of the 14, 12 selected 14.7, and gave reasons indicating that they understood why this was so: "There's a seven in the tenths place and a two in the tenths place"; "This has seven longs and this one only two longs": "I thought of money, 70¢ and 26¢". In the final interviews, held approximately ten weeks after the decimal unit, students continued to showed a much better understanding of decimal numbers. When asked to compare 7.3 and 7.29, 18 of the 20 students selected 7.3. Some students compared tenths or longs. Students who changed 7.3

to 7.30 seemed to understand that they were comparing thirty-hundredths with twenty-nine hundredths. The two who selected 7.29 were both confused by the fact that "7.29 has hundredths and 7.3 does not". Eighteen out of twenty also indentified 5.09 as smaller that 5.90, with explanations such as "...only 9-hundredths versus 90-hundredths" and "This is 9 small blocks, this is 9 longs." In another task, students were asked to order 0.72, 0.314, and 0.7. This was considered to be a transfer item, since the instructional lessons did not include ordering three numbers, nor did they include thousandths. Nine of ten students in one school were successful, but the teacher in this classroom had introduced thousandths on her own. Students at the other school had not encountered thousandths, and only three of the students were successful with this task.

The fence painting item was used again in interviews of 15 students immediately following the unit on fractions. The first comparison was between 3/6 and 3/4, and 14 of 15 answered correctly: "3/6 is only half, 3/4 is almost a whole"; "Fourths has bigger parts". When comparing 3/8 of the fence with 1/2 of the fence, 12 of 15 students answered correctly: "It would have to be 4 out of 8 to be half"; "4/8 is bigger that 3/8". In a second item, students were asked to sort ten fractions into three piles: those close to 0, those close to 1/2, those close to 1. Fourteen of the 15 students could correctly place all ten fractions. A third item asked students for a fraction between 1/4 and 2/4. This had not been covered in the instruction. Only four students were able to find one, either 1/3 or 3/8. But when asked to find a fraction between 1/4 and 1/2, six more students identified 1/3 as being between the two.

On the final interview, approximately five weeks after the fraction unit, 18 of the 20 students identified 2/3 as the larger of 2/3 and 2/5, with reasons such as "There are bigger pieces in 2/3"; "2/3 is more than a half. 2/5 is not half of 5 *(sic)*". Of the 20 students, 19 recognized that 1/2 was larger than 3/8: "To be equal, it has to be 4 instead of 3". Comparing 2/3 and 3/4 was much more difficult for the students. They had not had a problem this difficult in the instructional unit. However, 8 of the 20 selected 3/4 and made drawings or "pictured them shaded in my mind" to show they understood the problem. The most common incorrect answer was they they were equal because each "has just one piece left".

Discussion

This study was undertaken in the belief that the meaningful study of number magnitude concepts would lead to increased number sense. The high rate of retention illustrated in the final interviews indicates that children did understand fraction and decimal numbers sufficiently well to compare numbers in size. Students had somewhat more difficulty with fraction items than with decimal items. As mentioned earlier, this might be due to the fact that it is more difficult to formulate a small number of rules to follow when comparing fractions. Certainly, rules were formulated by students. For example, the rule "If the numerators are the same, then the fraction with the largest denominator is the smallest fraction" was formulated in each class at the conclusion of a lesson in which a large number of such cases were presented. However, no rule could be easily formulated for fractions where both the numerators and denominators were different.

The actual instructional time spent on number comparison was actually quite minimal. Each unit had several preliminary lessons introducing children to the manipulatives used in the units. It therefore seems that in classrooms where fractions and decimal numbers are taught meaningfully, with manipulatives, the small amount of additional time needed to teach number size concepts would be time well spent. Understanding of these concepts that should also assist students in later learning of computational estimation involving fractions and decimal numbers.

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FIRST GRADERS' UNDERSTANDING OF THE PRE-CONCEPTS OF NUMBER

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Plurality and position can be considered as preconcepts of natural number. In a study of the kindergartners' constuction of these two pre-concepts, some tasks related to the logico-physical abstraction of number were found to be quite difficult for them. In the present study, we interviewed 32 first graders using the same tasks. It was quite surprising to find that the same difficulties remained and that even with a higher success rate, at least half of the subjects had not yet constructed the invariants studied.

The idea of the pre-concepts of number stems from a model used for the analysis of this conceptual scheme developed by Herscovics and Bergeron (1988a). In this model two tiers are identified in the description of the understanding of a mathematical concept: the first tier dealing with the understanding of the physical pre-concepts, the second tier with the understanding of the emerging mathematical concept. Regarding the concept of number, two specific notions are viewed as pre-concepts of number: the notion of plurality (which distinguishes between one and many), and the notion of position of an element in an ordered set. Of course, when number is viewed as a measure of plurality and as a measure of position, the emerging mathematical concepts are those of cardinal number and ordinal number.

This paper deals solely with the two pre-concepts, plurality and position. According to the above mentioned model, one can identify three levels of understanding of these two physical notions. The first level, that of intuitive understanding, is based essentially on visual apprehension; it enables the child to compare two sets and to decide where there are many or few, if two sets are equal or not. It also enables the child to judge positional notions such as before, after, between, at the same time, first and last.

At the second level, that of logico-physical procedural understanding, the child can generate pluralities and ordered sets subject to the various constraints listed above. The generation of such sets is no longer approximate, but is based on a very accurate and precise procedure using one-to-one correspondences.

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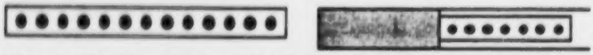
At the third level of understanding, that of logico-physical abstraction, the children perceive the invariance of plurality and position, whenever the given sets are subjected to various spatio-physical transformations such as elongation, dispersion, translation, partial hiding of the sets.

Using these criteria for the description of understanding, specific tasks have been designed and used with kindergartners in four Montreal schools. Results have been reported at last year's PME meeting in Veszprem, Hungary (Bergeron, J.C. & Herscovics, 1988; Herscovics & Bergeron, J.C., 1988b). In order to assess the evolution of these pre-concepts among young children, the same set of tasks was used with 32 first graders from four different schools in Greater Montreal.

Of course, all those tasks that were handled successfully by kindergartners were also handled easily by first graders, as could be expected. However, on the three tasks that proved to be difficult for kindergartners, the first graders' results were surprisingly low. The present paper will report the results obtained on these three tasks and compare them with the data obtained with kindergartners.

Invariance of plurality with respect to the visibility of objects

Children were given a row of 11 chips glued on a piece of cardboard. They were told: "Here is a large cardboard with little chips glued to it. Look, I'm putting the cardboard in a bag (the interviewer inserting the cardboard in a partially opaque plastic bag so that the three chips at the extreme left are no longer visible). And now, are there more chips in the bag, less chips, or the same number as before?".



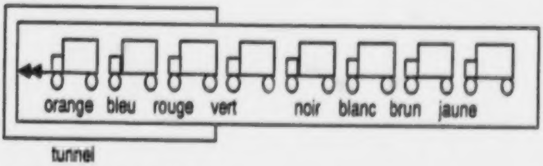
To the children who first answered that the number of chips in the bag had changed, the interviewer asked: "Are you telling me that there are now fewer (or more) chips in the bag (while moving her hand from one end of the bag to the other, in order to indicate that all the chips were to be considered)". Two children spontaneously said that there was the same number of chips as before, while 3 changed their answer, following the additional question, as exemplified by the comment of one little boy: "There is the same as before. There are some that we don't see so it means that we see less" The following table compares the first graders' success rate to that of the kindergartners:

	n	success rate
Kindergartners	30	4 (13.3%)
First graders	32	5 (15.6%)

Even at the end of the first year of schooling, the majority of students have not yet discovered the invariance of the plurality of a set when part of it is hidden from view. Of course, this does not mean that these children did not conserve **quotity**, that is, the ability to predict the number of chips in the bag, had they been counted before (Greco, 1962). For in fact, Herscovics & Bergeron, J.C. (1989) have established that 78% of the kindergartners did already conserve quotity. Somehow, we had expected that the disparity between the conservation of plurality and the conservation of quotity would have been resolved at the end of grade 1.

Invariance of position with respect to the visibility of the objects

A row of 9 little trucks was drawn on a cardboard, each truck coloured differently.



The children were told: " Look, here is a parade of trucks. Can you show me the green truck?" (in fourth position). After it was duly pointed out, the interviewer announced "The parade must now go under a tunnel" and then proceeded to slide the cardboard under a 'tunnel' so that the first three trucks were hidden. The children were then asked: a)"Do you think that the green truck has kept the same number in the parade?" After they had answered this question, they were asked : b) "Do you think that when the trucks are in the tunnel, this can change the number of the green truck ? "The above task was repeated with the row being moved up by the length of another three trucks (question c).

Even if for the two groups the rate of success for the invariance with respect to visual perception is greater for the concept of position than for the concept of plurality, the rate is still low, as shown in the following table:

	n	success rate for the questions		
		a	b	c
Kindergartners	30	16(53.3%)	not asked	15(50.0%)
First graders	32	9(28.1%)	18(56.3%)	17(53.1%)

A surprising result is that for question (a) the rate of success of the kindergartners is almost twice as high as for the first graders (53.3% vs 28.1%). A likely explanation might be that we had modified the material presentation of this task. Whereas in the preivious year we had used toy trucks with the kindergartners, we now used a strip of cardboard with cars drawn on it and a tunnel which completely prevented the subjects to see those cars underneath it. The reason for this change was that in the interviews with kindergartners we noticed that many children glanced at the cars inside the tunnel, thus jeopardizing the objectives of the task which aimed at evaluating the effect of having part of the row hidden from view.

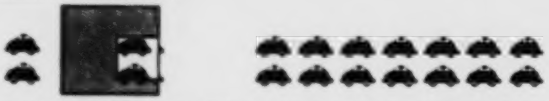
For question (b) we cannot compare the kindergartners with the first graders since this part was added only afterwards. We wanted to find out if a chance to reflect on a general property relative to the determination of the position of an object in a row would be different from a judgement made in the specific situation where the physical set up certainly influences the children's judgement. It is interesting to note that under these circumstances, twice as many subjects (56.3% vs 28.1%) were able to reach a right conclusion.

In the case of question (c), the rate of success for the first graders is almost identical as for question (b) (53.1% vs 56.3%). Two likely hypotheses can be invoked: on the one hand, the preceding reflection about the invariance of position when some objects are hidden might have helped them to overcome the cognitive obstacle induced by the hidden part; on the other hand, the improvement from question (a) might also be attributed to the fact that in question (c) the first three cars were visible, thus making it more evident that the hidden cars were still part of the row. In spite of the fact that they were not made to reflect on the general property (question b), the kindergartners obtained a rate of success just as high as the first graders. This is probably due to the fact invoked above, that is, the cars hidden underneath the tunnel could be perceived by looking alongside the row.

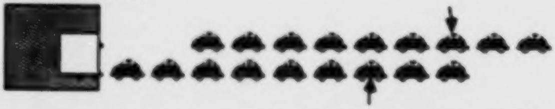
As was the case for the invariance of plurality, the children were also very dependent on visual perception of the objects for determining the position of an object in a row.

Invariance of position with respect to translation.

The interviewer aligned a row of 9 identical cars, and asked the children "Would you make a parade just like mine and next to it?" while handing over another 9 cars. Then using a blue colored sheet of paper (a river) and a small piece of cardboard to represent a ferry, she explained: "The parades must cross the river on a little ferry boat. But the ferry can only carry two cars at a time, one car from each parade. When we are ready, we take one car in my parade (putting her lead car on the ferry), and one car from your parade "(asking the children to put their lead car on the ferry). The ferry then crossed the river with the two cars, unloaded them, and came back for two more:



The cars were then put back in their initial position and the subjects were told: "Now I"m putting this little arrow on this car (the seventh car in the interviewer's row). Can you put this other arrow on the car in your parade which has the same number as mine?" Once this was done, the interviewer announced "Now look, the parades move on" while moving the child's parade a small distance and moving her own parade somewhat further by the length of two cars:



The children were asked : "Do you think that the two cars with the arrows will cross the river at the same time?" Following their answer, they were asked to show the interviewer how the two parades were to cross the river in order to verify that they were aware that the cars had to be ferried in pairs (see column 2). The interviewer then asked if the two cars marked by the little arrows still had the same number. Finally, they were asked if they thought that those two cars (now both in fifth position) would cross the river at the same time.

The following table presents the results obtained:

	N	will cross at the same time	underst. situation	still have same no	will cross at the same time
Kindergartners	30	10.0%	96.7%	26.7%	23.3%
First graders	32	21.9%	87.5%	62.5%	50.0%

At the kindergartners' level, it was found that very few children had constructed the invariance of position with respect to a translation. The first time the question was asked only 10.0% answered spontaneously that the two cars with the arrows would cross the river together. But when, following this they were asked if the two cars still had the same number, twice as many subjects (26.7%) answered affirmatively. This question probably induced an opportunity to reflect on the link between the position of the two cars and its invariance.This is reflected in the greater number of pupils believing that the two cars would cross at the same time, when asked the same question with the cars being in fifth position. It is also possible that a second factor operating here is that after the children had crossed two pairs of cars, the two marked cars were both closer to the river. So, we have here a combination of two factors: provoked reflection and visual perception. The fact that the great majority of these kindergartners has not yet perceived the invariance of position cannot be attributed to the lack of understanding of the task at hand. In fact, 96.7% of them have been able to manifest their understanding when they were asked to show the interviewer how the two parades were to cross the river.

The same discussion applies to the first graders except for the success rate that is approximatively double that of the kindergartners. However, a 50% success rate is quite low taking in consideration a full year of prior schooling.

CONCLUSIONS

The results show that even after a first year of formal schooling, the children still have problems with some aspects of the invariance of number, namely, the invariance of plurality and position with respect to the visibility of the objects, and the invariance of position with respect to translation. We see that the visual apprehension greatly affects the reasoning of children.

In fact, we saw that for the invariance of plurality with respect to visual perception, no difference was found between kindergartners and first graders, both having obtained a very low success rate of about 15%. For the invariance of position with respect to visual perception we found that about half the first graders had not discovered it. A comparison with the kindergartners could not be made because of a modification made in the physical materials used.

For the invariance of position with respect to translation we saw that the first graders had a success rate twice that of the kindergartners, but with still half the subjects not aware of this invariance. However, two things strike us here. First, the fact that having made the students reflect on the mathematical property ("do they still have the same number?") has probably contributed to their progress. Secondly, that learning has occurred so easily, with only one indirect intervention.

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The Role of Spatial Patterns in Number Development

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Based on clinical interviews with primary school children, six types of responses were identified to dot pattern tasks. A teaching experiment was conducted with one child to explore the use of dot patterns in number development. This paper discusses her use of counting methods and dot patterns and her progress during the teaching experiment. The importance of setting and context were identified as critical in the child's selection of method.

While Steffe, von Glasersfeld, Richards, and Cobb (1983) have developed a model of young children's counting which is quite useful, Hatano (1982) has proposed a mental regrouping strategy based on dot patterns as an alternative model of young children's number development. Apparently Japanese arithmetic instruction which makes extensive use of spatial dot patterns and little use of counting is effective (Easley, 1983). A better understanding of the potential of dot pattern use by children in constructing number and number operations is needed. Furthermore, imagery may play a prominent role in mathematical reasoning generally (Bishop, 1989; Presmeg, 1985; Skemp, 1987). The use of imagery and in particular, dot patterns in constructing number concepts, is not well understood. The goal of this research project was to explore children's number constructions as influenced by spatial dot patterns.

In many intellectual tasks, imagery plays an important role. As Johnson (1987) states, "Imagination is a pervasive structuring activity by means of which we achieve coherent, patterned, unified representations. It is indispensable for our ability to make sense of our experience, to find it meaningful" (p. 168). Kosslyn (1983) posited three conceptual acts involved in imaging. They are generating an image, inspecting the image (We will use the term re-presenting), and transforming the image. This view is compatible with what Bishop (1983) calls visual processing. Generating the image is a personal matter. Each person gives their own meaning to what they perceive. When a child views a briefly presented pattern of dots, some will see it as a collection to be counted and will construct a scan path for accomplishing the counting. Others will construct familiar subpatterns and determine the number by combining the numbers associated with them. Once an image has been constructed it does not remain in consciousness but must be re-presented when needed. In re-presenting an image, a person may view the image from a different perspective. The third aspect of imaging is transforming an image. In determining the number of dots in a spatial pattern, the objects may be mentally rearranged to form a familiar pattern for which the number is known. While it is impossible to directly observe the images a person constructs, the manner in which they use them can be inferred from their actions in problematic situations.

Spatial visualization involves more than mere perception of objects. It is not just a process of taking a mental picture and retrieving the picture from memory (Wheatley and Cobb, in press). As Lakoff (1987) states, "It is important to distinguish mental images from perception. A perception is rich in detail since our eyes are constantly scanning" (p. 444). "Different people, looking upon a situation, will notice different things. Our experience of seeing may depend very much on what we know about what we are looking at. And what we see is not necessarily what is there" (p. 129).

THE RESEARCH PLAN

There were two phases of this research. First, clinical interviews were conducted with 26 children ages 6-8 to explore their use of imagery in determining the number of dots in a collection briefly displayed. Secondly, an eight month teaching experiment was conducted with one child to investigate her use of spatial patterns on addition and subtraction tasks.

The dot pattern tasks

Arrangements of dots drawn on 5"x 8" cards as shown in Figure 1 were briefly shown to the child and he or she was asked "How many dots did you see?" and "'How did you see them?" If the child did not respond or seemed to be stymied, the card was briefly shown again. In some cases children described the arrangement and in other cases they drew a pattern to show what they had seen. The children were asked to explain their method of obtaining the number reported.

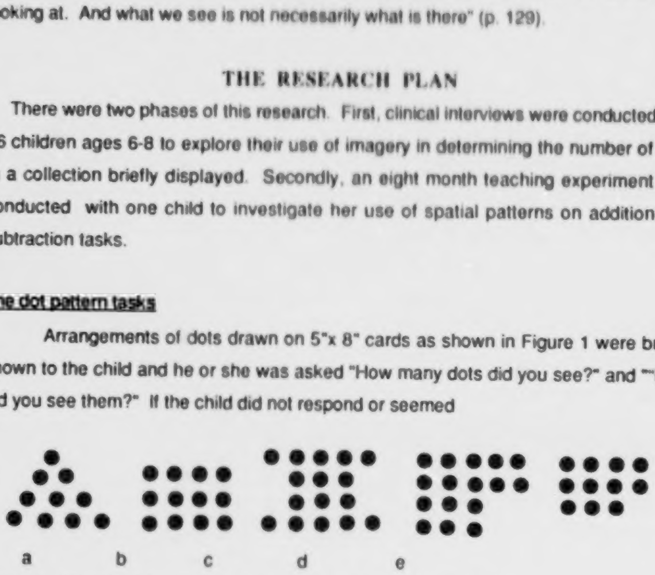

Figure 1. Sample dot pattern tasks used in the clinical interview

In a second dot pattern task, five cards were laid in front of the child and a card showing a pattern of dots was displayed for three seconds. The cards visible in front of the child had two, three, four, four, and five dots. After the card had been briefly shown the child was asked to find two of her cards that had the same number of dots as the one shown. The third dot pattern task was similar to the second except that a card showing two dot patterns separated by a vertical bar was shown and the child was asked to find one of her cards which had the same number of dots as on the card shown. The number of dots on the cards visible ranged from six to eleven.

Steffe, et. al. (1983) base their assessment of children's counting types in large part on responses to screened tasks, that is, tasks with some items not visible. If questions are posed with objects in view, little is learned about the child's potential for reasoning with numbers since she can just count each item by ones, touching them if necessary. In a similar manner, questions following presentations of dot pattern cards can provide information on which an explanation of the child's activity can be constructed. Dot patterns are useful because the child cannot work from visible arrangements of dots but must use self-constructed images. Further, the setting provides an opportunity for relating and transforming of images.

Dot pattern strategy types

Analysis of the video recordings of the clinical interviews with 26 children ages 6-8 resulted in the identification of six types of responses. They broadly fell into two categories, counting (types one and two) or transforming and comparing images (types three through six). These six types of responses are:

1. Counting while the card was being shown. Some children attempted to count the dots one by one. Because the exposure time of the card was short, this strategy was not effective; there was not time to count all the dots yet some children persisted with this method. There was no evidence they had constructed an image of the dot pattern. Once the card was no longer in view they stopped their counting activity even though all dots had not been counted.
2. Constructs an image and counts the dots using a re-presentation of the image. After the card had been shown, Sally pointed in the air as she counted the dots.
3. Constructs subpatterns. This strategy is illustrated by the child who reported, "I saw three, three, and two - - that's 3, 6, . . 7, 8."
4. Relates the pattern to a previous dot pattern. Some children determined the number of dots on the card shown by comparing their image to an image of a previous dot pattern. For example, Drew said, "That's the same as the other card. See, there was two rows of five and two rows of three. That makes 16."

5. Transforms an image. The child transformed the constructed image to a familiar arrangement for which the number was known. Adam transformed his image of a four-over-two pattern to a three-over-three pattern which he knew was six.

6. Constructs an image not shown for comparison. In this method the child "completes" a pattern. For example, shown two rows of four and a row of three, the child constructs an image of three rows of four and knows that is 12. Then determines the number in the arrangement presented by taking one away from 12. Drew said, "Eleven, "cause there was one missing."

The Teaching experiment

In order to investigate the use of dot patterns in number development, we selected a single child for study. Tammy was selected because her number development was below average for her age, she seemed to take tasks as problems, offered explanations of her activity, and had potential for use of spatial patterns. An eight month teaching experiment was conducted with that child. Clinical interviews were conducted in September, January, and April. The teaching experiment sessions were video recorded and held twice a week for one hour over a six month period. The tasks presented in the sessions varied from addition and subtraction to spatial tasks such as dot patterns, tangrams, drawing shapes seen briefly. Addition and subtraction tasks were presented in many settings. For example, using countable objects, paper and pencil computations, mental arithmetic, money problems, word problems, and games. Analyses were done after each session in order to give meaning to her actions and to design new tasks which would be problematic for her at the next meeting. However, in each session, we felt free to modify the proposed tasks based on our interpretation of her activity. Our goal was always to select tasks that would likely require a mental reorganization, that is, tasks for which her methods would no longer work. This practice allowed us to pose tasks which would provide information about her constructions.

At the time of the initial interview, Tammy was eight years ten months of age and in the third grade. The initial clinical interview revealed that Tammy could solve missing addend screened tasks (Steffe, et. al. , 1983) but had not internalized the number word sequence. Thus when she counted she was "in the action" and could not reflect on her counting activity. She did not spontaneously use thinking strategies. In determining the number of objects in two collections, she made extensive use of well developed finger patterns. To find 7 + 2, she extended seven fingers, five on one hand and two on the other all in one motion, and then put up two fingers all at once and recognized the nine finger pattern formed. For 6 + 5 she used her fingers to count on from six. Being "in the action" of counting, she would have to repeat this procedure if asked the same question a few minutes later. When asked to add 17 + 6 she made 17 marks and 6 marks on a paper and counted them. At times she attempted to use a vertical algorithm but became confused and abandoned the attempt. For Tammy, an addition task was a signal to count, using finger patterns if possible. While she was slow and showed little reflection on addition and subtraction tasks, she was able to determine quickly the number of dots in a spatial pattern. On the dot pattern tasks, her action exemplified type three (above) in which subpatterns were formed. For example, she explained her method by saying, " You have two and two that's four. Then you have three here and that's seven." While Tammy was quite good at determining the number of dots when a pattern was presented briefly, she rarely related the pattern to a previous pattern or transformed her image to a familiar one. Tammy could represent her self-constructed images but rarely transformed them.

DISCUSSION

Throughout the teaching experiment, Tammy had a strong inclination to count in an unreflective manner in determining the number in two collections. Her rare use of dot patterns in finding sums may have resulted from her intention to count rather than use images. In order to find sums by combining dot patterns, it would be necessary to transform the re-presented images, i.,e., mentally rearrange the dots. We conjecture that she had well developed static imagery but poorly developed dynamic imagery (Piaget and Inhelder, 1971). For example, she had great difficulty with tasks which required mental rotation of images.

During the first five sessions she began to solve addition problems with addends greater than 10 by reusing her fingers but she had difficulty with subtraction problems. During the fifth session we became aware of her difficulties in using a counting back strategy. Thus in the next session we asked her to count back from 26 and found that she could not do it. In attempting to count back she made errors and even paused 15 seconds between saying twenty-three and twenty-two. We also found that Tammy could not count by two's. Even after several practice sessions, she counted "2, 4, 8, 9, 10." in counting 10 objects.

In order to investigate her use of dot patterns and number combinations, we played a Domino Ten Game with Tammy in the eleventh session in which two dominoes could be played end to end if the sum was ten (a double nine set was used) . The Domino game was played for two and half hours over four sessions. In this setting Tammy consistently used a counting on strategy to test combinations and showed no recognition of the number based on the dot patterns. Even when there was a seven pattern showing she would count by ones to find how many. She did not combine dot patterns to find the number of dots on two dominoes as she had done on the dot pattern cards - her intention was to count. Her play, although slow, was intelligent and she rarely missed a combination. She was also aware of the play of others, frequently commenting on strategy and possible moves of others.

We conjecture that with the dots on the dominoes visible, Tammy could systematically use primitive counting to test combinations and had no inclination to develop or use other methods. There was no recognition of combinations which made ten, she relied entirely on a counting method. Although the setting was conducive for use of dots patterns, and we knew she could use dot patterns, Tammy's context precluded use of them. Because she was in the action of counting and did not reflect on the counting activity, she did not reorganize her schemas and develop an efficient, number pattern based method.

Several changes in Tammy's mathematical activity were observed during the teaching experiment. On December 5, the fourteenth session, Tammy used a compensation thinking strategy in solving 9 + 3 as a subproblem to a computation task. She wrote 10 and 2 and said twelve. In subsequent sessions she had many opportunities to use thinking strategies but on only one other occasion was use of a thinking strategy identified. Over the six months of the teaching experiment, there was a noticeable decrease in the explicit use of finger patterns. On January 23 she solved 8 + 6 by making a fist and looking at it. She obtained the answer in four seconds, most likely using a curtailed counting on strategy. On the same day she was given 5 plus 3 followed by several other similar tasks and then asked 3 plus 5. Her response was, " You already asked me that one!" This was considered as evidence that she was reflecting on her actions.

CONCLUSION

In interpreting students' actions in number settings, it is important to recognize the role played by context. It is possible that students may not succeed as well as they might because they operate in a context which limits their progress. For certain tasks Tammy operated in a counting context and performed less well than when she used spatial patterns. Tammy's mathematical activity was context bound. Some settings triggered a counting response and in other settings she would use dot patterns. However, she lacked flexibility and rarely changed context to use alternative methods which might be more effective. Perhaps if teachers use a variety of settings students will be more likely to develop flexible methods for thinking about numbers. Use of dot patterns may encourage some students to develop powerful methods.

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SYMBOLIC REPRESENTATION OF ADDITION AND SUBTRACTION WORD PROBLEMS: NUMBER SENTENCE ERRORS

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This paper presents a discussion of the typical number sentence errors that occur when first-graders are asked to symbolically represent addition and subtraction word problems. Two instructional treat- ments are presented and the persistence of certain types of errors are noted.

Several studies have documented the performance of children in writing number sentences to symbolically represent various types of addition and subtraction word problems (Bebout, in press; Carpenter, Hiebert, & Moser, 1983; Carpenter, Moser, & Bebout, 1988; DeCorte & Verschaffel, 1983; Feiyu & Shanghe, 1988; Ishida, 1988; Lindvall & Ibarra, 1980; Moser & Carpenter, 1982). Children's success with symbolic representations appears to vary according to the correspondence between the structure of the word problem and the number sentence formats that children know. For example, when asked to write number sentences for simple word problems, i.e., join, separate, and combine addition problems, children are very successful because the structures of these problems correspond to the standard, or canonical ($A + B = \square$ and $A - B = \square$), sentence forms that are familiar to children; when asked to write sentences for more complex problem types, such as missing addend or compare problems, children are less successful because the structures of these problems do not correspond directly to the familiar canonical forms. Many of these unsuccessful attempts, however, indicate potentially successful number sentence forms.

For example, consider a typical missing addend problem:

> Polly has 7 cookies.
> Her brother gave her more cookies.
> Then she had 11 cookies.
> How many cookies did her brother give her?

Prior to instruction some children are successful in overcoming the problem's additive structure and attending instead to the solution's subtraction structure; they write canonical number sentences of the form $11 - 7 = \boxed{4}$. Other children appear to keep their focus on the problem's additive structure and to ascertain the correct solution; they represent this structure with sentences of the forms

$7 + 11 = \boxed{4}$ or $7 + 4 = \boxed{11}$. And yet another group appear to extract the given numbers, to determine the operation, and to carry out a solution bound to the incorrect sentence; they write a sentence of the form $7 + 11 = \boxed{18}$. This latter sentence error appears to be a persistent type that has been documented in later assessments (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981).

Ishida (1988) categorized children's incorrect number sentences into the following five error types: Type A $7 + 11 = \boxed{4}$: An inappropriate sentence with a correct solution; Type B $7 + 4 = \boxed{11}$: An appropriate sentence with the wrong numeral identified as the correct solution; Type C $7 + 11 = \boxed{18}$: An inappropriate sentence with an incorrect solution; Type D: No attempt to write a sentence; and Type E: All other incorrect sentence errors.

This paper presents the types of number sentence errors written by two groups of children who were given different instructional treatments. One group was taught to write only canonical number sentences that represented the basic number fact corresponding to the solution (Ishida, 1988); the other group was taught to write either canonical or noncanonical number sentences that represented the structure of the problem (Bebout, in press). Although the results of the treatments are not statistically comparable because of treatment and evaluation differences (problem type, problem order, problem inclusion, duration of instruction, and number domains), the pattern of error types that appeared postinstructionally indicated children's developing expertise in symbolically representing word problems.

METHOD

Samples

Two populations of children were studied. Group J, the Japanese children, consisted of 137 first-graders in four classrooms. Group A, the American children, consisted of 46 first-graders in two classrooms.

Instructional Treatments

Children in Group J received their regular curriculum over the academic year. Briefly, this curriculum included instruction on writing canonical forms for the following problem types: Join and combine addition, separate and compare, three element addition and subtraction, and combine and substitution subtraction (Ishida, personal communication). Children in Group J were not shown noncanonical forms; this is included in the second grade curriculum.

Children in Group A received a special 14 session instructional treatment over a five week period during the spring of first-grade. Briefly, this treatment included instruction on writing forms that corresponded to the following problem types: join and separate, combine addition and subtraction, join and separate change unknowns, and join and separate start unknowns (see Bebout, in press). Children in Group A were shown both canonical and noncanonical forms.

Instruments of Evaluation

The instruments of evaluation were group word problem tests administered before and after instruction. In these tests children were asked to write a number sentence for and to solve several types of addition and subtraction word problems. The problem types, the order of administration, and the problem number domains differed for the two groups: Group J children were given join, separate, combine addition, combine subtraction, compare, and missing addend problems with low number domains; Group A children were given join, separate, combine addition, combine subtraction, missing addend, three other change and start unknown problems, compare, and equalize problems with high number domains. The intersection of types common to both studies were the six problems used in the Group J study.

The incorrect forms of number sentences were categorized into the five error types suggested by Ishida (1988): Type A errors included inappropriate sentences with correct solutions; Type B errors included appropriate sentences with correct solutions, but with the wrong numeral identified as the solution; Type C errors included inappropriate sentences with incorrect solutions; Type D errors included the lack of attempt to write a sentence; and Type E errors included all other error types.

RESULTS

The success of children in both groups to symbolically represent six types of word problems before and after instruction are presented in Table 1. For the join, separate, and combine addition problems, most of the children in both groups were successful in writing correct sentences after instruction. Their performances on the three other problem types, the combine subtraction, compare, and missing addend problems, were less successful.

(Insert Table 1)

Data pertaining to children's errors on the combine subtraction, compare, and missing addend problems both before and after instruction are displayed for Group J in Table 2 and for Group A in Table 3. These error data are arranged according to preinstructional error type on the missing addend problem and are presented in levels. Children at Level 1 made no errors and wrote correct canonical number sentences; children at Level 2 wrote sentences with Type A or B errors; and children at Level 3 committed Type C, D, or E errors.

(Insert Tables 2 & 3)

The Level 1 children in both groups were very successful in writing sentences for all problem types after instruction. The Level 2 children in both groups had a lower number of Type C, D, and E errors and also were more successful after instruction.

The Level 3 children improved also but to a lesser extent than then other levels. Their performances on the non- instructed problem for each group exhibited a pattern, or progression, of improvement from the most severe Types C, D, and E error types to those that are very close to correct sentence forms, Types A and B. On the compare problem, the errors of Group A children changed from predominently Type C and D errors to a substantial number of Type A errors. On the missing addend problem, the errors of the Group J children showed this same pattern too, but with a larger number of Type B errors.

By better understanding and anticipating these types of errors, instruction can planned that will help children at all levels to symbolically represent word problems. A further discussion will be provided during the presentation.

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Table 1
Number and Percent of Correct Preinstructional and Postinstructional Number Sentences on Word Problem Types for Groups A and J

Word Problem Type	Group A (n = 46)		Group J (n = 137)	
	Pre	Post	Pre	Post
Join	30 (2) 66.7%	44 (4) 97.8%	129 (4) 92.4%	136 (1) 99.3%
Separate	32 (3) 71.1%	41 (5) 91.1%	120 87.6%	133 (2) 97.1%
Combine (addition)	31 68.9%	41 91.1%	129 94.2%	136 (2) 99.3%
Combine (subtraction)	10 22.2%	35 (1) 77.8%	68 (12) 49.6%	112 (4) 81.8%
Compare	4 8.9%	5 11.1%	82 (1) 59.9%	118 86.1%
Missing Addend	3 6.7%	39 (1) 86.7%	20 14.6%	70 (1) 51.1 %

(The number in parentheses indicates the number of calculation errors within the correct sentence total.)

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Table 2
Number and Percent of Preinstructional and Postinstructional Number Sentence Error Types for Group J Children at Different Levels (n = 137)

Word Problem Type	Number Sentence Errors									
	Type A		Type B		Type C		Type D		Type E	
	pre	post	pre	post	pre	post	pre	post	pre	post
Combine (subtraction)										
Level 1[a]	1 1%(d)	0	0	0	1 1%	2 2%	1 1%	0	0	0
Level 2[b]	4 3%	1 1%	13 10%	1 1%	2 2%	1 1%	2 2%	0	3 2%	1 1%
Level 3[c]	4 3%	4 3%	3 2%	2 2%	13 10%	12 9%	15 11%	0	3 2%	0
Compare										
Level 1	0	0	0	0	2 2%	0	0	0	0	0
Level 2	6 4%	2 2%	4 3%	0	3 2%	1 1%	1 1%	0	1 1%	0
Level 3	6 4%	2 2%	0	2 2%	14 10%	10 7%	13 10%	0	2 2%	0
Missing Addend										
Level 1	0	2 2%	0	1 1%	0	0	0	0	0	0
Level 2	13 10%	3 2%	32 23%	12 9%	0	5 4%	0	0	0	0
Level 3	0	7 5%	0	9 7%	31 23%	22 16%	31 23%	0	0	0

a) for Level 1, n = 3; b) for Level 2, n = 12; c) for Level 3, n = 31
d) to nearest whole percent

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Table 3
Number and Percent of Preinstructional and Postinstructional Number Sentence Error Types for Group A Children at Different Levels (n = 46)

Word Problem Type	Number Sentence Errors									
	Type A		Type B		Type C		Type D		Type E	
	pre	post	pre	post	pre	post	pre	post	pre	post
Combine (subtraction)										
Level 1[a]	0	0	0	0	1 2%(d)	0	0	0	0	0
Level 2[b]	1 2%	1 2%	1 2%	0	4 9%	1 2%	2 4%	1 2%	0	0
Level 3[c]	0	3 7%	0	1 2%	18 39%	3 7%	6 13%	0	2 4%	2 4%
Compare										
Level 1	0	0	0	0	1 2%	0	0	1 2%	0	0
Level 2	2 4%	3 7%	3 7%	2 4%	2 4%	3 7%	4 9%	0	0	1 2%
Level 3	0	12 26%	1 2%	0	21 46%	11 24%	6 13%	0	2 4%	6 13%
Missing Addend										
Level 1	0	0	0	0	0	0	0	0	0	0
Level 2	2 4%	1 2%	10 22%	0	0	1 2%	0	0	0	0
Level 3	0	2 4%	0	0	20 44%	1 2%	6 13%	0	5 11%	0

a) for Level 1, n = 3; b) for Level 2, n = 12; c) for Level 3, n = 31
d) to nearest whole percent

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GEOMETRY, MEASUREMENT, AND SPATIAL VISUALIZATION

CASE STUDIES OF CHILDREN'S UNDERSTANDING OF THE CONCEPT OF LENGTH AND ITS MEASURE

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This paper reports the initial results of a study aimed at determining the understanding of the concept of length among 9 year-old children. Various tasks have been developed on the basis of a two-tier model of understanding. Regarding the first tier, results indicate that the logico-physical understanding of length is relatively well achieved. On the other hand, at a second tier, the understanding of the 'measure of length', the study reveals many difficulties, especially those relating to the understanding of the unit of measure and its representation on a ruler.

The importance of the concept of measure in the school curriculum does not have to be stressed. The learning of the concept of length is fundamental since it is the first step that children undertake at the mathematical level in their acquisition of the more general notion of the measure of magnitudes. These last few years, several researchers have studied problems related to the understanding and the learning of the measurement of length. Carpenter et al. (1980) have noted that large numbers of 9 year-olds can fail some tasks when the context of the situation is varied ever so slightly. This reveals a superficial understanding of the basic concepts. According to Hart (1981), similar difficulties are still found extensively among secondary school students, thus indicating their persistence. Some researchers have dealt with the specific problems involved in the learning of this concept. Hiebert (1984) has brought out some difficulties that first graders have in understanding the relations between the choice of unit and the measure resulting from it. Similarly, Bessot and Eberhard (1983) have examined the links that 7 and 8 year-olds can establish between measure and the marks on a ruler.

The objective of the present investigation is to contribute to the study of these problems. It aims to describe and classify the cognitive obstacles that children encounter in their construction of the notion of length. The originality of this research is that it does not simply aim at establishing a list of all the difficulties but rather, it attempts to look at these in a conceptual framework enabling us to get a better grasp of the children's construction processes. Such a conceptual framework was was established prior to the present research (Héraud, 1989). It built on the Extended Model of Understanding developed by Herscovics & Bergeron (1988). This model suggests an important distinction between , on one hand, **logico-physical understanding** which deals with physical objects and the spatio-physical transformations on these objects, and, on the other hand, **logico-mathematical understanding** resulting from reflection on the procedures and actions pertaining to mathematical objects. For the topic at hand, it enables us to distinguish three levels of understanding of the physical concept of length and three components of understanding of the emerging mathematical concept: the measure of length

The study we intend to present deals with the first phase of a larger project concerning the clinical observations of the child's construction of the notion of length. Using the above model, we have designed a sequence of tasks that characterize the different levels and components of understanding.

UNDERSTANDING THE PHYSICAL CONCEPT OF LENGTH

Referring to the previously mentioned model, at the first tier of understanding, length is considered as a physical magnitude prior to any numerical quantification. We can distinguish here three levels: **intuitive understanding** that results from a form of thinking based essentially on the visual perception of length, **logico-physical procedural understanding** relates to the acquisition of logico-physical procedures verifying intuition, and **logico-physical abstraction** which refers to the construction of logico-physical invariants.

We have sought to determine the children's knowledge with respect to these three levels of understanding. A test was developed and presented in the form of individual interviews with 25 children from the same third grade in an urban neighborhood school (average age 9 years and 1 month). On the basis of their school performance, these children could be divided equally into 5 groups: strong (S), Strong-Average (SA), Average (A), Average-Weak (AW), and Weak (W).

Intuitive and procedural understanding

To first assess intuitive and procedural understanding, the child was given a bunch of six straws of nearly equal lengths, measuring between 5.5 cm to 7 cm, with an average difference of 3 mm, and was asked to arrange these according to size. Our results show that the large majority of the children we have tested (92%) have no difficulty with this task, despite the fact that the lengths of the straws were very close to each other. Children proceeded by using a comparative measure, that is, by comparing directly a chosen straw with another one and then arranging them in the proper order. Only 2 subjects in this class were unable to handle this task (one W and one AW). It thus seems that intuitive and procedural understanding of length, at the logico-physical level, is fairly widely achieved by 9 year-olds.

Logico-physical abstraction

On the other hand, results are far less positive at the logico-physical level of abstraction. We have used as criterion the child's perception of the invariance of the length of an object with respect to various figural transformations, prompted in this by some of the well known tasks developed by Piaget and his colleagues (1948/1973). At first, we assessed the invariance with respect to **orientation**: two identical straws were positioned perpendicular to each other. The child had to decide whether or not they were the same length. Moreover, assuming that the straws were licorice, they had to indicate whether or not there would be the same 'amount to eat'. To assess the invariance with respect to **fragmentation**, two identical straws were chosen and one of these was then cut in two and placed under and parallel to the other one. The same questions introduced earlier were repeated. Finally, the two aspects, **orientation and fragmentation**, were considered jointly, one of the two straws was cut into 5 parts and laid out in a non-rectilineal 'path'.

The results obtained on these three tasks show that a non negligible proportion of children (20%) encounter real problems at this level. Remarkably, the same five subjects had difficulties with each of the three problems, thus indicating the persistence of their perception regarding these three types of invariance. For each of these children (with one exception), the error in judging the question of length was confirmed by the same error on the question of licorice, which indicates that the mistakes were conceptual and not due to some lack of understanding of the wording.

For those who succeeded on the three tasks, some facts are worth bringing out. For instance, with respect to the change in orientation, some of them took care to distinguish the terms that were used, stating that one was higher than the other and the other one was wider, but that it amounted to the same thing since it was possible to place one in the position of the other; this clearly indicated an awareness of reversibility. A similar note applies to the fragmentation in two parts. Nearly half the children claimed that the two straws were of the same length, for all one had to do was glue together the two pieces to get back the initial one. However, it is of interest to note that several children stated spontaneously that the two straws, the complete one and the cut up one, were not of the same length, which might have been construed as an error had they not specified later on that they were taking into account the space separating the two cut up pieces; this indicated that, contrary to initial appearances, the problem was indeed understood. The results obtained on the third task (orientation and fragmentation) confirm the preceding data, with several children indicating that it may look longer with the five cut up pieces but that it amounts to the same if one does not take into account the space between them.

One child even mentioned that in the case of the licorice he was quite sure that it was the same length but that he neveretheless preferred the one with "the little pieces".

Regarding logico-physical abstraction, one can conclude that it seems to be achieved by a large majority of 9 year-olds. However, it is important to dissociate the understanding of terminology from that of the concept itself for the word 'length' can be confused with those of 'width' and 'height'.

UNDERSTANDING THE CONCEPT OF THE MEASURE OF LENGTH

Referring to the Herscovics & Bergeron model, we can identify a second tier of understanding. We then no longer view length as a physical magnitude, but rather as a quantifiable magnitude that can be expressed numerically, that is, from the perspective of measure. This tier consists in three components: **logico-mathematical procedural understanding** (the acquisition of logico-mathematical procedures) as well as **logico-mathematical abstraction** (the construction of invariants), and **formalization** which refers, among other things, to the rational use of certain forms of mathematical symbolization.

In order to assess the extent of the children's understanding of the measure of length concept, we selected 11 children who had succeeded in the previous tasks and who represented 4 different levels of mathematical ability ranging from Strong to Average-Weak. We opted for semi-standardized interviews using a list of questions prepared in advance while leaving the possibility for the investigator to change the wording in case a question had not been understood. These interviews were video-recorded for later analysis.

Logico-mathematical procedural understanding

The objective here is to determine the extent to which the child uses the notion of **unit**, notion that constitutes the core of any measuring operation. To assess this we have developed two tasks that cover two important aspects of this concept.

The choice of identical units

Children were presented with two "trains" consisting of little rods, each one having a total length of 26 cm but including a different number of rods (5 for the first one and 8 for the second one). The two trains were separated and slightly offset, this in order to prevent any comparative measure. Subjects were then asked to compare the lengths of the trains without moving them closer together. We wanted to determine if they were not confusing the notion of length with that of the number of units and if they might feel the need for a common unit of reference.

Spontaneously, 4 of the 11 children indicated that the train with the most rods was the longest , which shows that they were focusing on the number of units without taking their size into account. Five children indicated that they did perceive the important role of the size of the units when they expressed that even if there were more "cars" in the second train, it didn't mean that it was longer than the first since there were "little cars in the second and large ones in the first". Only one child used a purely visual comparison without taking into account the number of units.

On the other hand, even if the subjects were aware that it was not easy to compare lengths on the basis of different units, only one of them thought about using a same reference unit in order to compare the two trains. This indicates that the choice of identical units does not appear as an evident need to these children.

Iteration of the unit

The suggested task consisted in finding the length of a path drawn on a sheet by using a paper clip. This object was chosen in accordance with a similar task appearing in the second NAEP in which only about half the 9 year-olds could provide a correct answer (J.Hiebert, 1981) .

With the children in our test, the main problem was carrying forward the measuring unit with more or less precision. Indeed, it is not easy for a child to put a mark at the exact end of the paper clip and to start again with precision from the same spot. This explains why only 3 out of the 11 children provided a correct response (7 paper clips). The others, who all had some problem in moving the clip forward, gave answers that were approximations such as "7 and a little bit". One single child had serious problems since he was using his finger to mark off the carrying forward point and this gave him 6 as an answer.

Logico-mathematical abstraction

The objective here is to determine if children are aware of the **invariance** of the measure of length with respect to various figural transformations; it is also to see to what extent they perceive the links between length, considered as an invariant physical entity, and its measure, which can vary according to the unit of measure selected. Two different tasks were thus prepared.

Comparison of measures with respect to displacement

With a first task, we wanted to determine the extent to which children had remained dependent on the figural context, with or without the presence of pre-determined units. Thus, initially, the pupils were given two strips of different length (12 cm and 9 cm), slightly offset with respect to each other (see fig.1), they had to show what had to be added to the shorter one (by cutting up a strip of paper) to make it as long as the other one.

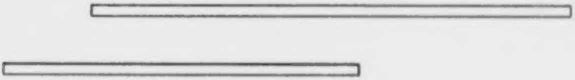

Figure 1

Following this, a similar task was suggested, consisting of two horizontal lines (10 cm and 8 cm) drawn on a sheet, but this time with some marks indicating units (of 2 cm) on each of the straight lines (see fig.2).

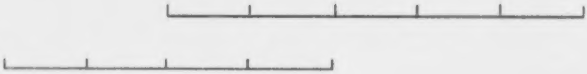

Figure 2

Results indicate that 5 of the 11 children succeeded on the two tasks, by basing themselves on visual estimation in the first instance, and on the number of units in the second one. For the other 6, it is interesting to note that the first task was generally successful while for the second one, several pupils remained at a perception level of the unit. For three of them, only one single unit was added but without taking into account the length of this unit. Also noteworthy is the erroneous procedure consisting of adding to the lower line the part on the left that extends beyond the first line; it was used systematically by two subjects in each of the tasks.

The variability of measure with respect to the size of the unit

The aim here was to determine if children could recognize and use the simple ratios existing between two types of units in order to deduce the corresponding measures. The following problem was presented: a 30 cm segment was measured by the children with 6 cm rods and they were then asked to predict the result if smaller rods were to be used (2 cm and then 3 cm).

Results are rather positive. Among the 11 children, 8 of them used ratios; 7 did find the correct ratios with 2 of them (classed as Strong) making a direct transfer for the corresponding measure (of the type 3 x 5 and 2 x 5); the other 5 found the answer by repeated addition and this can be attributed to their lack of familiarity with multiplication.

Formalization

One of the main aspects of this last component of understanding is the appropriate use of **conventional measuring units**. When linked with utilization of a **ruler**, this formalizes the notions acquired previously. Although we knew that the selected children did not, in all likelyhood, possess all the preliminary notions needed for a correct utilization of a ruler, nevertheless, we wanted to take a closer look at the major problems that they faced in this context. To achieve this, the children were to use the rulers (in centimeters) shown in Figure 3.

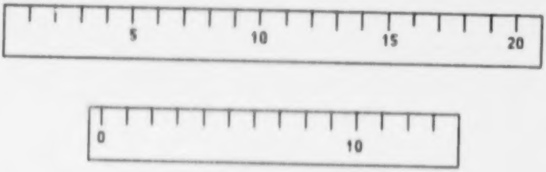

Figure 3

The use of the first ruler involved added difficulties caused by the fact that the mark for 0 was not shown and that the last extremity exceeded 20 by one unit. For the second ruler, the 0 mark was somewhat off the left extremity of the ruler. With the first ruler, children had to measure a 24 cm segment, thus longer than the ruler, which required moving it forward (case 1), and also a 14 cm segment (case 2). With the second ruler they had to measure a 19 cm segment (case 3) and another one 13 cm long (case 4).

The best results were obviously obtained in case 2 since all that was needed was a direct reading. Thus 8 of the 11 children found 14 as a result. Regarding the 3 pupils who did not succeed, it is noteworthy that they found 15 as an answer for they positioned the 1 mark at the end of the segment and not the 0 mark which was not visible.

With regards to case 4 for which a direct reading was also sufficient, the results were clearly poorer, since 5 children who were successful in case 2 now were failing. They were positioning the end of the ruler and not the 0 mark accross from the end of the segment they were measuring. This could be viewed as lack of attention but it is doubtful since the same mistake was found again in case 3 where the same problem occurs when the ruler is moved forward. Moreover, it should be pointed out that the three children who failed case 2 succeeded "logically" in case 4 since they were measuring from the first mark on the ruler.

Regarding cases 1 and 3, in which the ruler had to be moved forward, these created many problems for the children. Only one single pupil did solve correctly both cases. In case 1, not a single child wanted to mark off 20, but 5 of them indicated 20 when in fact it was at 21 cm, and 2 others who, after moving forward the ruler found that the end of the segment was in front of the 5 mark, stated that it measured 25 (cm). These two facts show that children tend to focus on the numbering appearing on the ruler without looking for its meaning. Moreover, 2 children did not move forward the ruler relying instead on a simple estimation of the part remaining to be measured.

CONCLUSION

The results of our investigation indicate that intuitive and procedural understanding of the logico-physical concept of length is well mastered by most 9 year-old children. However, logico-physical abstraction is far from acquired, implying that its construction occurs somewhat later. At the second tier of understanding, the difficulties are more evident, especially those concerning the understanding of unit. Thus, in terms of procedural understanding, some children think of measure in terms of the number of units, regardless of the unit size. Regarding logico-mathematical abstraction, the necessity of keeping a unit of length constant is not always apprehended. Finally, in terms of formalization, important difficulties are often encountered regarding the proper use of a ruler, such as associating marks on a ruler with units.

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RELATIONSHIP BETWEEN SPATIAL ABILITY AND MATHEMATICS KNOWLEDGE

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In this study differences in mathematical knowledge between fifth grade children of high and low spatial ability were examined. The Wheatley Spatial Ability Test (WSAT) was administered to fifth grade students (N=54). Based on the results of this test, six girls were chosen for clinical interviews because they had either high or low spatial scores. All interviews were video recorded for transcription and analysis. Mathematical knowledge was assessed using tasks such as conservation of area, linear measure, concept of one fourth, proportional reasoning, multiplicative reasoning, and solving a nonroutine problem. The results of the spatial tasks closely paralleled those of the WSAT. Analysis of mathematical tasks revealed that while the low spatial girls did well in school mathematics, their knowledge was instrumental. The high spatial girls mathematics, however, was more relational.

There is a long history of interest in the relationship between spatial ability and mathematical knowledge (Bishop, 1980, 1989; Clements, 1982). The results of this research, however, are by no means clear. Guay and McDaniel (1977), for instance, using tasks of simple and complex spatial abilities, found a positive relationship between mathematics achievement and spatial abilities in elementary school children. In studies with older (six through twelfth grade) children, Fennema and Sherman (1977, 1978) found significant correlations between spatial visualization and all levels of mathematics achievement. In a study of engineering students, Lean and Clements (1981) found that more visual students performed more poorly on mathematical tests that students who processed information by verbal-logical means.

Much contradictory evidence is the result of differing ways of assessing spatial ability and mathematical knowledge. In this paper we shall accept the position of Kosslyn (1983), who identified three distinct conceptual acts involved in imaging: Generating an image, re-presenting an image and transforming an image. The generation of an image is personal. Each person gives meaning to what he or she perceives. Once an image has been constructed it does not remain in consciousness but may be re-presented when needed. Upon re-presentation, this image can be transformed. By accepting this definition, we have eliminated static imagery from consideration and will examine only examples of dynamic imagery (Piaget and Inhelder, 1971).

There is little reason to expect spatial reasoning to be related to instrumental mathematics. Relational mathematics (Skemp, 1987), however, because it has been constructed meaningfully by the child, may be inherently spatial. The purpose of this experiment was to examine differences in mathematical knowledge between children with high and low spatial ability.

METHODS

Selection of subjects. The Wheatley Spatial Ability Test (Wheatley, 1978) was administered to two classes of fifth grade students in a local elementary school. This test is a 100-item pencil and paper test which was administered to all students in a class simultaneously. In this test, students are given a sample figure and must then decide if five congruent figures may be matched by rotation of the sample figure.

In order to discourage students from using analytical instead of spatial methods, an 8-minute time limit was imposed. Students' test scores were computed using the formula:

number correct - 1/2 (number incorrect)

A perfect score was 100. Since responses on the test were dichotomous, a student could be expected to obtain a score of 25 by chance alone.

In all, the test was administered to 54 students. Scores ranged from almost perfect (98.5) to well below chance (-6.5). This wide range of scores prompted questions about differences in mathematical knowledge between students with high and low spatial scores.

Based on an analysis of the spatial test results, six children with extreme scores (highest 10% and lowest 10%), were selected for further study. Because of differences in the mean scores between boys and girls, and the underrepresentation of boys in the extreme groups, this study was restricted to girls. Of the six girls chosen, three had the highest test scores and three scored near or below chance.

Procedures. After selection, the six girls were given a series of individual clinical interviews to further explore their spatial reasoning, cognitive level, and mathematical knowledge. The clinical interview was chosen as the research technique because of its potential for revealing the nature of children's spatial and arithmetic reasoning in a way not possible with standard format tests. All interviews were video recorded for later transcription and analysis.

Tasks used to assess spatial ability and mathematical knowledge. Several tasks with concrete materials were used to assess arithmetic knowledge. In a test of multiplication understanding, students were initially given 36 multilink cubes and asked to make a rectangular region. They were then asked about the relationship between the dimensions of the rectangle, the number of cubes in the rectangle, and their relation to a multiplication problem. They were then asked to make rectangular regions of different dimensions with the same number of cubes. Finally, if they were successful with small numbers of cubes, they were given 91 cubes and asked to construct a rectangular region with them. In a test of division knowledge, students were given 36 unifix cubes ans asked to share them fairly among four people. Students were also given one nonroutine problem to solve (Steffe, 1988). They were first given 27 unifix cubes and asked how many piles of three could be made. The interviewer then asked the question, "Suppose I gave you some more cubes, so that you had 36 in all. How many piles of three could you make?"

Students were also given three tasks of cognitive development. As a test of proportional reasoning, students were given a modification of the paper clip problem (Karplus, Karplus, & Wollman, 1974). The test of conservation of area was that used by Piaget, Inhelder, and Szeminska (1960). In a final test (Flake, 1978), the students were given three square pieces of paper and asked to fold each into four equal parts in a different way and color one part of each; having done done this, they were asked if the parts are the same size or different and why.

Two additional tasks of spatial ability were used, one involved the use of concrete materials (Davidson and Willcutt, 1983) and the other, a computer program, "Transform" (Flake, McClintock and Turner, in press). In the concrete materials task, the subjects were presented with two sets of grids upon which a Cuisenaire rod pattern could be imposed. They were asked if the patterns were flipped, turned (90 or 180) or were different. The subjects had access to two sets of rods, so they could use a number of strategies to solve the problem. In the computer program "Transform", two shapes were presented on a video screen. The students' task was to decide if the shapes could be matched using the available transformations to move one to cover the other. The transformations available to the students were: slide, turn, flip and dilate.

RESULTS

Tests of mathematical knowledge. The most striking differences between the high and low spatial girls was revealed by the multiplication task. Although all the low spatial girls knew their multiplication facts, their performance on the rectangle task revealed some striking deficits. All used rather primitive, trial and error, methods of forming the rectangle. One student, Laura, was unable to form any rectangle with 36 cubes, after starting several open rectangles and stating that she does not have enough cubes to make one, she finally was able to construct a 7x5 rectangle with 35 cubes, but she called it a 7x4. When asked about the facts 7x4 and 7x5, she knew the correct answers, but saw no relationship between the answers and the rectangle she had just constructed. Kimberly, another low spatial girl, performed similarly in many ways. She was able to form a 6x6 rectangle, but called it a 6x4x16. Again, when quizzed about facts, she gave the correct answer, but saw no relationship to the rectangle problem. The third low spatial girl, Helen, correctly identified her rectangle and realized that the number of cubes was the product of the sides. She thought that there was only one possible rectangle which could be made with thirty-six cubes.

In contrast, the high spatial girls all went about the task quite systematically. Two of them made multiple rectangles easily with the thirty-six cubes, correctly identified the dimensions, and understood their relationship to multiplication. Karen enjoyed the activity, particularly constructing a 2 x18. One of these girls, Amy, did not know her multiplication facts, but had developed her own strategies for determining products that she did not know.

In the division tasks, two of the low spatial students made errors when sharing fairly. Helen counted only one pile and miscounted. Laura, however, counted all four piles obtaining the result nine, nine, eight, and ten. She saw no problem with this answer. In the nonroutine problem, all the girls were able to get nine piles of three. Laura, at first, guessed 21, but when prompted, made the piles and counted. In the next part of the task, the high spatial students were able to verbalize a logical solution to the problem. The low spatial girls were not able to verbalize a method, even though one of them obtained the correct answer.

These tasks also offered insight into the girls counting strategies. The low spatial girls tended to count cubes singly by pointing at each one. The high spatial girls were much more likely to count by two's or three's. At one point, Karen was given three ten-cube chunks and a six-cube chunk. She counted the six-cube rod and one ten-cube rod by two's, then compared the height of all the ten-cube chunks to make sure they were the same and said, "36".

Tests of cognitive level. None of the students were successful at applying formal proportional reasoning to the Mr. Short - Mr. Tall problem. Of the high spatial girls, Tiffany tried to use a visual estimate, while Karen used addition. None of the low spatial girls, however, thought the problem was solvable. The results of the conservation of area task were less marked. Only two of the girls, one high and one low, gave conservation responses. Another high spatial girl, however, responded in a unique way which could not be considered indicative of either.

In the concept of one-fourth task, none of the low spatial girls were able to find more than two ways to fold the squares into fourths, and only one was able to see that the area of the regions was equal. The other two very confidently picked a region which was larger. In contrast, all the high spatial girls were able to find three ways to fold the paper, even though they required some experimentation to do so. Two of them also recognized the equivalence of the regions.

Spatial tasks. The results of two spatial tasks closely paralleled the result of the WSAT. All three of the high spatial girls were able to solve the Cuisenaire rod problems without the use of concrete materials, while none of the low spatial girls could. One low spatial girl, Kimberly, gave incorrect answers even after she had used the rods to construct the patterns. In their performance on the computer program, all the girls required some time to learn the mechanics of the program. After this initial learning, however, two distinct strategies developed. The high spatial girls performed the appropriate transformations on the shape and then slid it to match, soon became very efficient and able to finish in a short period of time. In contrast, the low spatial girls slid the shape first and then tried to match it with other transformations. They never appeared to adopt on algorithm, but tried things at random, often repeatedly, without success. As a result, they took much longer to complete the task. One girl, Laura, was never able to match the shapes successfully.

DISCUSSION

The results of this experiment indicate that spatial ability as measured by the WSAT is a good predictor of mathematical knowledge. One of the low spatial girls (Kimberly) had a very high I.Q. and performed well in school mathematics, but her performance on the tasks of meaningful mathematics, was much like the performance of the other low spatial girls. In contrast, one of the high spatial girls was performing rather poorly in school mathematics, but had an excellent grasp of mathematical ideas and could find creative solutions to problems.

The low spatial girls performance on instrumental tasks was above or near average, which is not surprising since there is little reason to believe that spatial reasoning is related to instrumental mathematics understanding. The learning of rules does not require the construction, re-presentation or transformation of any images. Relational understanding, however, would seem to require at least the construction and re-presentation of images, and in most cases, transformation of images. Consider, for example, the multiplication tasks with multilink cubes. In order for the students to easily build a rectangular region, they had to have an image of a rectangular region. The high spatial girls' approach was quite systematic, suggesting they were working from an image, while the low spatial girls seemed to work at random until they found a pattern which fit the definition of a rectangle. Likewise, the task used by Steffe (1988) is much more easily accomplished if students are able to visualize piles of three in 27 and 36. In constructing any mathematical relationship, whether it is the inverse relationship between multiplication and division, or a geometric one such as in the golden ratio, perhaps some form of imagery is involved.

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REFLECTION , POINT SYMMETRY AND LOGO

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Reflection and point symmetry are transformations taught in the secondary shcool in France (pupils 11 - 15 years old) which both involve a problem of orientation. This research has the purpose of making the properties related to orientation apparent for the pupils in the learning of reflection and point symmetry. For this we undertook the construction of sequences in a LOGO environment where pupils are induced to determine the reflection or the symmetry of an angle.

I.INTRODUCTION,PROBLEMATICAL CONTEXT.

Reflection and point symmetry are two transformations which are studied at school (first and second year of secondary school in France) for which it is useful to define the image of an angle. These two transformations on a plane are in fact restrictions to a plane of corresponding spatial transformations but have not the same effect on orientation as these spatial transformations. They, also, correspond to phenomena in every day life, the mirror reflection for the reflection, the beam of rays of light converging on a point and then separating for the point symmetry. It seems necessary that in teaching these two transformations the problems of orientation should be taken into account. For this we choose to give to the pupils taks in which they will be obliged to determine the image of an angle.

We have therefore produced a teaching programme on reflection and another on point symmetry using the LOGO language on a micro computer: LOGO with the following restrictive list of primitives: FD, BK, RT, LT, ORIGINE and commands necessary to erase lines comprises a microworld in which in order to determine the symmetry of a figure made up of segments, the pupils have to determine the symmetry of an angle by expressing its measure and its orientation. Generally in classrooms, rules and compasses are the only essential instruments available for geometry. The only instrument related to angles is the protractor- seldom used or mastered by the pupils and which in any case does not take orientation into account (1985 LABORDE).

The choosen microworld makes it possible for angles to be taken into account without formalisation. In the first and second years of secondary school under consideration here, all formalisation concerning angles was previously out of the question. In addition the microworld makes the pupil draw the symmetries of the figures thus producing the contour without calculating each vertex. This encourages a global vision of the figure and its symmetrical aspect (1987 GALLOU-DUMIEL).

II.STUDY OF THE COMPUTER ENVIRONMENT

1) Comparison betwen the ways of drawing in the two environments under considerations.

The turtle on the screen has the same role as the point of the pencil on the paper. However there are two major differences: firstly the turtle indicates the direction and the orientation while a pencil does not and secondly in the computer environment there is not a direct access to all the points of the sheet.

2) Ways of drawing in the computer microworld.

The pupils do not have the means in terms of knowledge and consequently of "procedures" to find the image of a point under certains transformations by calculating its coordinates.

A "procedure" consisting of drawing all the images of the vertices of the figure and joining them is very easy to use in a pencil / paper environment because there is a direct access to all points of the drawing sheet. This "procedure" is not economical in the choosen computer environment, on account of the limitations due to this environment: it would be constantly necessary to move the turtle back to the vertices of the given figure. The pupils in the computer microworld, are induced to use the following "procedure" to draw a polygon : they put the turtle in a vertex, make it turn in advance to the direction and the sense of a side, make it move forward a length equal to that of the side, turn it again to put it on the direction of next side, and so on ... In the choosen environment, on the screen, the triangle "turtle" represents both, a point, a direction and an orientation (1981, ROUCHIER). Not only are the pupils required to draw a continuous line but they also must fully explain the properties related to the orientation of angles. There is a two-fold intermediary. That of a repertory of actions expressed in a language and that of the computer environment. Significant effects can be expected from this two-fold intermediary.

III.EXPERIMENTS

1)Variables of the tasks and choice of the figures.

We decided to choose for the pupils tasks of construction of symmetries of different figures. To choose the figures we first study the variables they are the characteristics of the figures, a modification of which may involve the pupils changing their way for drawing the symmetry of a figure.

If, for example, the centre of the symmetry and the turtle are both positionned on a vertex of the given figure, pupils will always begin the construction of the symmetry of the figure by making the turtle advance in the direction of a side closed to this vertex and in the opposite sens.

If the centre of the symmetry is outside of the figure it is possible that pupils may draw sides of the symmetry of the figure which are not parallel to the corresponding sides on the given figure and it is also possible they draw them parallel and in the same sense of the sides of the given figure, realising a reflection instead of a point symmetry.

If the centre of the symmetry is inside the figure, pupils sometimes use "procedures" related to a "conception" (1988 GALLOU-DUMIEL) in which the given figure and the image cannot be intersected.

For example one "conception" is that the difference of orientation between reflection and point symmetry does not exist. Another is that the symmetry of a figure cannot be intersected with the given figure.

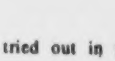
We can also see the coexistence of several "conceptions" in the mind of pupils. Once the list of variables was drawn up we chose the figures in the following way: for the first figure, all "procedures" give an exact result, for the later figures choices of the values of the variables are successively made so that alnost all possible combinations appear. The strategy to favour the use of "procedures" which conduce to false result: the pupils will have to modify their "procedures" and their "conceptions" in realizing other tries with the same figures.

2)Experiments

Each of the sequences was tried out in two classes. Pupils have to make a new try in case of false result and this obliges them to change their strategies and conduces to an evolution of their conceptions. Exercices and corrections are put in the computer in langage LOGO. Similar sequences were tried out in two other classes in a paper-pencil environment. Concluding tests in a paper/pencil environment were administrated in all the classes.

IV. EVALUATION

The comparison between the pupils carrying out the sequences in the computer environment and in the paper/pencil one shows both a different number of errors and different types of error. If we take different types of error first for the reflection in a paper /pencil environment there is an important error, that we call error of indefferenciation between the axes of reflection and the figure. For the given figure where D is the axis of reflection



the error of indifferentiation corresponds to three different drawings:

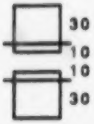


fig 1

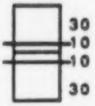


fig 2

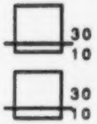
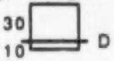


fig 3

When pupils produce the third drawing (fig 3) they both use the "conception" in which axis of the reflection is a part of the figure and the "conception" in which reflection is not differentiated from translation.

The error of indifferentiation does not exist in the computer environment. The errors for the given figure



are:

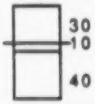


fig 4

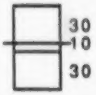


fig 5

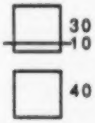
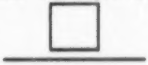


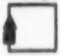
fig 6

None of them correspond to the "conception" in which axis of reflection is a part of the figure because the axis is not reproducted on the image. The three drawnings correspond with the "conception" in which the given figure and the image one cannot be intersected. We are sure that in the second drawning (fig 5) there is not the "conception" in which the reflection is not differentiated from translation because the image of the square is not a square. The third drawing (fig 6) can correspond to the conception in which the reflection is not differentiated from translation because the drawing is similar with the one which has be done for a previous figure:

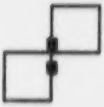


and the axis of reflection is not taken into account.

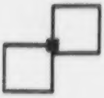
We have similar errors for point symmetry with CARREF :



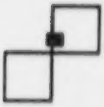
The errors in paper-pencil environment are:



error 1

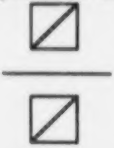


error 2

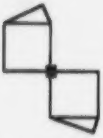


error 3

In the choosen computer environment pupils only do errors 2 and 3.
For the errors of orientation we make the following observations.
For CARREB which is the second figure of the sequence on reflection the following error is done by 80% of pupils in all the environments:



For MAISONF witch is the nineth figure of sequence on point symmetry the following error is done by 25% of pupils in the computer environment and 40% in paper-pencil environment:



We see that for MAISONF there are less pupils who make the error than for CARREB.
It indicates that a learning took place before doing MAISONF for the pupils and it shows us the difference between learning of the consideretec notion in computer environment and in paper-pencil environment. As for us the different number of errors is concerned, it was found that errors linked to orientation are similar at the beginning of the session but they decrease more rapidly in the microcomputer environment than in the paper/pencil one. One of the classes carried out the session on point symmetry in the computer environment after having done the session on reflection the previous year. At the begining of the session there was a much lower proportion of errors. For the final tests the pupils who worked in the micro - computer environment have a higher success rate then those who worked in the paper/pencil environment and especially, they worked with more precision and exactness.

The difference between the two groups of pupils is also noticeable in their recognition of the presence or absence of the axis of the reflection. This was done better in the group who worked in the micro - computer environment.

Similar results are found in the case of the centre of the symmetry but the difference is slightly less noticeable.

These results show that the two sequences in the computer environment develop the following capacities:

1) the global vision of the figures and their symmetrical aspect ;

2) the differentiation between given figure / element of symmetry and the symmetry of the figure.

The comparative study of the "procedures" (1988 GALLOU-DUMIEL) used by the pupils during sessions, allows us to understand something of the learning "procedure" and some of the causes of error. In this comparative study the computer environment using LOGO is a didactic tool.

V. CONCLUSION

The drawing "procedure" required by the computer environment using a restricted list of LOGO instructions would appear to be a tool for teaching those aspects of geometry in which the notions of angles and orientation play an important role. This has been found to be fruitful in teaching reflection and point symmetry. This type of study should be extended to other fields of geometry teaching.

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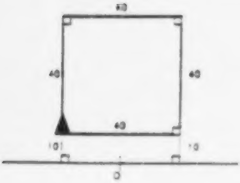
1981 ROUCHIER J., *Problèmes, procédures, programmes étudiés et réalisés par des enfants de CM2 utilisant un micro-ordinateur*, Revue française de pédagogie, 56, 18-26

CARREA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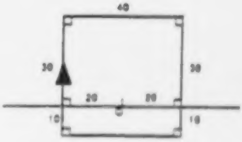
O

CARREC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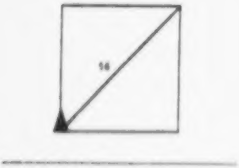
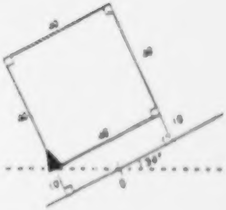
CARREB



CARRED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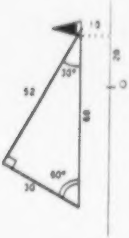
TRIANGLEA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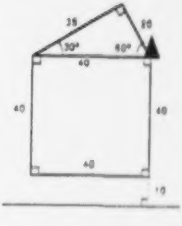
60
60°
30

MAISONA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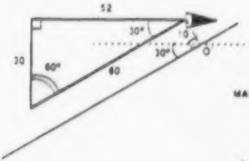
TRIANGLEB


52
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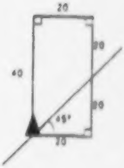
RECTANGLEA


20
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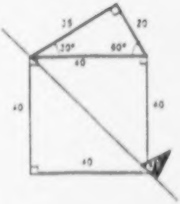
MAISONB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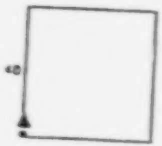
CARREE



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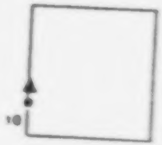
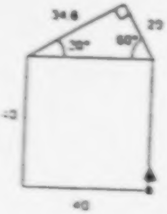


MAISONF


34.8
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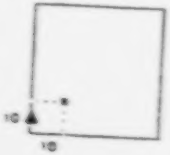
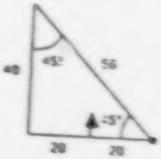
CARREG



TRIANGLEC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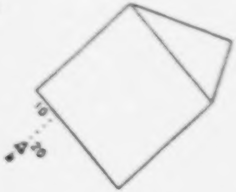
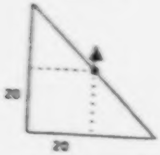
MAISONG



TRIANGLED



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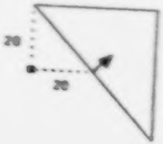

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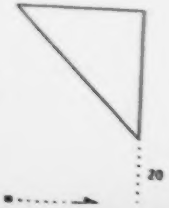
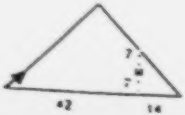
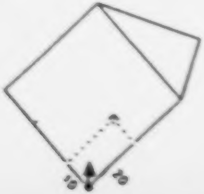


TRIANGLEG





MAISONH

Variables of point symmetry	Variable position of the centre of symmetry					Variable turtle				Variable figure	
	on the contour of the figure	in a vertex	in the middle of a segment	inside	outside	placed in the centre of symmetry	in a vertex of the figure	in the middle of a segment	parallel or perpendicular to a segment	number of segments	presence of angles other than those of 90°
CARREE	X	X				X	X		X	4	
CARREF	X					X			X	4	
CARREG				X			X		X	4	
TRIANGLEC	X	X						X	X	3	X
TRIANGLED	X		X			X		X	X	3	X
TRIANGLEE					X			X	X	3	X
TRIANGLEF					X				X	3	X
TRIANGLEG				X			X		X	3	X
MAISONF	X	X				X	X		X	6	X
MAISONG					X	X			X	6	X
MAISONH				X			X			6	X

X when the answer is yes

BEST COPY AVAILABLE

STORY EDITING AND DIAGNOSIS OF GEOMETRY UNDERSTANDING

George W. Bright
University of Houston

This study investigated the effects of debriefing on performance at editing a story about squares to a story about either circles or equilateral triangles. Nineteen teachers participated; 10 did circle editing before debriefing and triangle editing after, while 9 did the reverse. More correct changes were made (a) after debriefing than before and (b) for circle editing than triangle editing. More analysis is needed to determine if errors reflect deep misconceptions or superficial difficulty with terminology.

A series of studies has been designed to investigate story editing as a tool for diagnosing understanding of geometry concepts among preservice and inservice teachers. Both mathematically incorrect explicit changes and failures to change wording that leave incorrect concepts in the story have been coded and analyzed.

In an earlier study (Bright, 1988) preservice and inservice teachers first individually edited a story about a family of squares to a story about equilateral triangles and then reedited the same story in groups of two or three. In both individual and group settings, superficial changes were made more consistently than substantive changes. Group editing produced more changes than individual editing, though the subjects were also more experienced during the group exercise. Some changes were not made in the stories that should have been; for example, references to "diagonals" were left in, even though triangles do not have diagonals. This study attempted to determine if experience at editing, along with feedback and discussion on the types of changes that needed to be made, might help focus attention on substantive content rather than surface features.

METHOD

The story is shown below; changes had to be written in by hand.

A Really Square Tale

Once upon a time there was a family of squares: Sam Square, Sara Square, Jimmy Square, and Janey Square. They lived in a condominium

> complex named Square One. Sam had four sides and very sharp corners. His wife Sara had four angles and very straight sides. Their diagonals met in the middle, a fact not unusual for squares. Their children, Jimmy and Janey, sometimes tilted to the left, so they were confused with the children of the Diamond family.
>
> One day in January it snowed, and the hill outside the Square's condo became very slick. Jimmy and Janey wanted to go out and play with their sled, but Sara was afraid they would get hurt. After all, squares don't roll down hills very easily. "You're so square, Mom!!" whined Jimmy. After much begging and pleading, however, Jimmy and Janey were allowed to go out and slide down the hill.
>
> On the fourth trip, Jimmy fell off and broke one of his diagonals in two, right at the mid-point. Janey got out the glue and tried to patch him up, but she succeeded only in breaking his other diagonal, half-way between the mid-point and a vertex. Sam packed Jimmy up in a box, and mailed him to the Square Repair Shop.
>
> They didn't hear from him again for four weeks and four days. By that time he had lost a lot of weight; in fact, one of his sides had fallen off, along with the two attached angles, and he was beginning to look like a triangle. Sara quickly cooked up his favorite meal - burger squares; and with a lot of tender loving care, he returned to his true shape: four sides of equal length, four right angles, sharp corners (after his father), and straight sides (after his mother).

The changes of primary interest were name-of-figure changes (e.g., "a family of squares"), property changes (e.g., number of sides, relationships between diagonals), other mathematical changes (e.g., rotation of a square makes is look like a diamond), and consistency changes (e.g., last names).

Subjects

One class (N = 19) of teachers in a course on diagnosis participated. One half of the class (N=10) was given 15 minutes to edit the story individually from squares to equilateral triangles. The stories were collected and subjects were debriefed. Then subjects were asked to reedit the story individually from squares to circles. These stories were collected and subjects were dismissed. The other half of the class (N=9) edited the story from squares to circles, were debriefed, and then edited the story from squares to equilateral triangles.

Procedures

At the beginning of the period, students were asked to participate voluntarily in a study designed to try out a method for finding out what students know about geometry. Students were told that they might find this technique useful later in their own classes. The subjects were split randomly into two groups; one group was then taken to an adjoining classroom; the author and a colleague were in charge of the two groups.

Students in each group were told the following: You will be given a story about a family of squares. The story contains some information about squares and their properties. Your job is to change the story so that it becomes a story about equilateral triangles (or circles). Your new story should contain information that is accurate for equilateral triangles (or circles). The stories were passed out, and students edited individually for 15 minutes. The edited stories were then collected.

The students were then debriefed. Students were asked to recall from memory what changes they had made. As each change was presented, it was labeled (e.g., property change) by the researcher in charge of that group. Any category not represented by their changes was explicitly mentioned, with an example given of that category. The debriefing lasted no more than 15 minutes.

Students were then told the following: You will be given the same story about a family of squares. The story contains some information about squares and their properties. Your job is to change the story so that it becomes a story about circles (or equilateral triangles). As before, your new story should contain information that is accurate for circles (or equilateral triangles). The stories were passed out, and students edited individually for 15 minutes. The edited stories were then collected and the students were dismissed.

RESULTS

Edited stories were scored as follows. At each potential change point a tally was made according to whether a change had been made. The changes were counted according to type: (a) name changes, (b) property changes, (c) other relationship changes, and (d) consistency changes. (See Tables 1 and 2.)

Table 1

Tallies of Changes

Type of change	Performance pattern [a]	Counts [b]			
Name	4, 0, 0	7	(7)	5	(4)
	3, 1, 0	2	(2)	5	(5)
	2, 2, 0	0	(0)	0	(1)
Property	7, 0, 0	1	(2)	0	(2)
	6, 1, 0	4	(1)	3	(5)
	5, 1, 1	1	(0)	0	(0)
	5, 2, 0	1	(2)	2	(0)
	4, 3, 0	0	(4)	4	(2)
	3, 4, 0	1	(0)	1	(0)
	2, 5, 0	1	(0)	0	(1)
Other	3, 0	8	(8)	6	(6)
	2, 1	1	(1)	3	(2)
	1, 2	0	(0)	1	(2)
Consistency	11, 0	2	(4)	2	(6)
	10, 1	2	(4)	5	(2)
	9, 2	0	(0)	3	(0)
	8, 3	3	(1)	0	(2)
	7, 4	2	(0)	0	(0)

[a] A, B, C = A correct, B incorrect, C skipped
A, B = A correct, B skipped

[b] A (B) C (D) = A students editing circles first, (B) triangles second,
C students editing triangles first, (D) circles second

Table 2

Average Number of Correct Responses

Type of Change	Circles		Triangles	
	First	Second	First	Second
Name [a]	3.7	3.3	3.5	3.8
Property [b]	5.1	5.4	4.7	5.1
Other [c]	2.9	2.4	2.5	2.9
Consistency [d]	8.9	10.2	9.9	10.2

[a] number of possible changes = 4 [b] number of possible changes = 7

[c] number of possible changes = 3 [d] number of possible changes = 11

Each group of students generally made more correct changes after the debriefing than before, especially property and consistency changes. On the first editing, only 1 of the 19 subjects made all property changes correctly; 7 more made six of the seven changes correctly. On the second editing, 4 subjects made all property changes correctly; 6 more made six changes correctly. On the first editing, 4 subjects made all consistency changes correctly; on the second editing, 10 subjects did so.

Both before and after the debriefing, more correct property changes were made for editing from squares to circles than from squares to equilateral triangles. As in the earlier study, students made a higher percentage of superficial changes (name and consistency changes) than substantive changes.

Some of the understandings and misunderstandings of these subjects are exhibited in the correct and incorrect changes that they made. For the circle editings, one student deleted references to"diagonals" before the debriefing, while four students did so after the debriefing; two students changed diagonals to circumference before the debriefing, while none did after; and two students left in references to diagonals before the debriefing, while none did after. Both before and after the debriefing, two students speculated that radii met in the middle. For the triangle editings, five students deleted references to diagonals before the debriefing, while none did after the debriefing; five students changed diagonals to sides before debriefing, while two did after; four students changed diagonals to angles before debriefing, while one did after; but four students changed diagonal to altitude or "hypotenuse" after the debriefing, while none did before. (Perhaps they were trying too hard to find another term!)

In terms of changes for the angles, the most popular changes for triangle editings were to indicate that the angles were equal or 60°. Three students indicated that the triangle had 3 right angles before debriefing, while one student did so after debriefing.

DISCUSSION

The increase in changes after the debriefing is not surprising. The activity was unusual, so the opportunity during the debriefing to see "modeling" of what was expected probably contributed to the increase. That circle editing was easier was also not a big surprise. Changes for triangle editing required a greater depth of understanding, in that properties of triangles had to be carefully related to each other. Changes for circle editing could be dealt with by thinking of "diagonal" as "diameter." Given the typically incomplete backgrounds of elementary teachers about geometry, however, it is not surprising that subjects had difficulty. Too, if one assumes that the subjects were relatively mathematically naive then one would expect that they would have difficulty relating properties.

The story editing activity allowed students to demonstrate proactive use of mathematics rather than reactive use as is typically called for in a diagnostic test. For example, this difference is exhibited in the failure of students to change "diagonal" to something more appropriate, especially in the editing to triangles. The data show that misunderstandings can be either passive or active; students can explicitly write something that is wrong (Radii meet in the middle.) or they can simply ignore incorrect statements created by changing the name of the shape (Their diagonals met in the middle, a fact not unusual for triangles.).

That there were so many different replacements for the term, "diagonals," was somewhat surprising; such a wide range of changes was not observed in the earlier study. More investigation seems warranted about whether these options represent significant conceptual misunderstandings or more minor difficulties with vocabulary. It may also be important to relate students' editings to the van Hiele levels of geometric thought.

Many of the misunderstandings might fairly easily be observed with traditional diagnostic techniques; for example, responses to test items. However, it seems much more difficult to find out whether these students really "believe" that diagonals exist in triangles through such traditional means. Both active and passive misunderstandings can lead to situations in which incorrect concepts might be learned. We need to become at least as aware of the effects of passive misunderstandings as we are of the effects of active misunderstandings.

Story editing seems to hold promise as a diagnostic tool; there may be misunderstandings that are exhibited in this setting that are not easily exhibited elsewhere. However, a correlation needs to be made between performance in story editing and in other diagnostic situations (e.g., diagnostic test, clinical interview). Do students exhibit the same (mis)understandings across all situations? If not, what are the factors that influence performance in each type of situation? Too, students may view story editing more as a writing exercise than a mathematical one, and consequently math anxiety may not interfere with performance in this situation as much as in other (more test-like) situations. It is unlikely, however, that all mathematics content is equally amenable to story construction. Those areas that might best be used need to be identified, and stories need to be created to highlight important conceptual problems that students are likely to encounter.

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THE VAN HIELE MODEL OF GEOMETRIC UNDERSTANDING AND GEOMETRIC MISCONCEPTIONS IN GIFTED SIXTH THROUGH EIGHTH GRADERS

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This study investigated the thinking in geometry according to van Hiele's five levels of 43 students in the sixth through eighth grades who had been identified as gifted. Analysis of both clinical interviews and paper-and-pencil tasks indicated that the majority of subjects are Level 0 thinkers who consistently recognize shapes by their appearance as a whole but 23% of the subjects exhibited thought patterns characteristic of a higher level without having achieved at least one of the previous levels. Specific geometric misconceptions were also identified.

Dutch educators P. M. van Hiele and Dina van Hiele-Geldof proposed a linearly-ordered model of geometric understanding. The van Hiele theory asserts that there exist five hierarchical levels of geometric thinking that a successful learner passes through: Basic Level - visualization, Level 1 - analysis, Level 2 - abstraction, Level 3 - deduction, and Level 4 - rigor. According to the van Hieles' model, the learner cannot achieve one level without passing through the previous levels. Progress from one level to the next is more dependent on educational experiences than on age or maturation, and certain types of experiences can facilitate (or impede) progress within a level and to a higher level (Fuys, 1984).

Previous research tends to support the hierarchical nature of the van Hiele levels within several populations. Joanne Mayberry (1981) found sufficient evidence among 19 undergraduate preservice elementary teachers to support this aspect of the theory, but she rejected the hypothesis that an individual demonstrated the same level of thinking in all areas of geometry included in the school program. The van Hiele levels of her subjects were quite low: they did not recognize squares as rectangles and did not perceive relationships between classes of figures.

In examining high school sophomores, Usiskin (1982) found that over 80% of these students can be assigned a van Hiele level be means of a paper-and-pencil test, but students may be in transition between levels and therefore difficult to classify. Burger and Shaughnessy (1986) found mainly Level 0 thinking for subjects in grades K-8. They described the levels as dynamic rather than static and more continuous than discrete. Fuys, Geddes and Tischler (1985) utilized instructional modules in geometry with sixteen sixth graders and sixteen ninth graders. They found entry levels of 0 and 1, but several students, especially those deemed above average in mathematics ability prior to instruction, exhibited Level 2 behavior by the completion of the six hours of clinical interviews and instruction. They also reported several misconceptions or errors found among these sixth and ninth graders. Among the examples cited were thinking "sides" refers only to vertical segments, using the phrase "straight lines" when referring to parallel lines, and thinking that a parallelogram has to have oblique angles (Fuys, Geddes and Tischler, 1985, p. 199). Hershkowitz (1987) found several geometric misconceptions displayed by students in grades 5 through 8. Examples include misidentification of right triangles, isosceles triangles, quadrilaterals and altitudes in various types of triangles.

Research indicated that gifted, average, and retarded children all follow the same pattern of progression through the Piagetian stages (Roeper, 1978; Weisz & Zigler, 1979; Carter & Ormrod, 1982). Gifted students showed superiority on Piagetian tasks over students of normal intelligence at every age level tested. Piaget proposed that the transition to formal operational stage occurs at ages 11 to 12. Carter and Ormrod (1982) found that the majority of subjects of average intelligence were still transitional to formal operations even as late as age 15. They also found that the gifted subjects entered formal operations successfully by 12-13 year of age (p. 114). Does the gifted students' ability to operate abstractly earlier than other students affect the linearly ordered development hypothesized by the van Hieles?

The purpose of this study was to investigate the thinking in geometry according to van Hiele's five levels of subjects in the sixth through eighth grades who had been identified as gifted and to identify specific geometric misconceptions held by these students.

METHOD

Subjects

The present study focuses on the levels of geometric understanding and misconceptions among students in the sixth through eighth grades who have been identified as gifted based on IQ or standardized test scores and teacher recommendations. The population consists of two distinct groups of subjects: the first group includes seven seventh graders and six eighth graders who were enrolled in an Algebra I course in a small rural district. All subjects in this group had scored above 420 on the Mathematics Subtest of the Scholastic Aptitude Test which was taken when they were in sixth grade. The second population representing 27 different school districts consisted of 15 sixth graders, 10 seventh graders, and 5 eighth graders attending a one week summer camp for the Academically Talented.

Procedure

The van Hiele level of the first groups of subjects was determined using a 25 item multiple choice paper-and-pencil test developed by the Cognitive Development and Achievement in Secondary School Geometry Project (CDASSGP) (Usiskin, 1982). In addition, they participated in a 30 - 45 minute interview based on Mayberry's questions. The summer camp subjects completed the CDASSGP test as well. Additionally, selected questions of particular interest from the interview were administered in written form to these students. The paper-and-pencil and interview questions focused on the concepts of square, isosceles triangle, right triangle, circle, parallel lines, similarity, and congruence.

RESULTS

Test Scores

The distribution of the CSASSGP test scores by group can be seen in Table 1. A number of the unclassifiable scores (labelled ? in the tables) were related to students showing mastery of Level 4 type problems when they had not mastered Level 3.

The distribution of highest van Hiele levels mastered by grade level appears in Table 2 below.

Table 1

% of Subjects by Group at Each van Hiele Level
as Determined by the CDASSGP Test

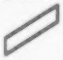
		van Hiele Level						
Group	n	below 0	0	1	2	3	4	?
Algebra	13	0	62	15	8	0	0	15
Camp	30	10	43	13	7	0	0	27

Table 2

% of Subjects by Grade Level at Each van Hiele Level
as Determined by the CDASSGP Test

		van Hiele Level						
Grade	n	below 0	0	1	2	3	4	?
Eighth	11	0	55	9	18	0	0	18
Seventh	16	6	56	13	6	0	0	19
Sixth	16	13	38	19	0	0	0	31
Total	43	7	49	14	7	0	0	23

Interviews

Analysis of the protocols from the interviews indicates several patterns not apparent in the multiple choice tests. All subjects identified a square with its sides rotated 45 degrees from parallel to the sides of the paper as being a square, but 60% of the subjects did not identify such a rotated rectangle as being a rectangle. 7% did identify  as being a rectangle. The three students who did this included "has four right angles" or "has four 90 degree angles" as part of their definition of a rectangle. However, the four right angles did not appear to be the attribute they were focusing on. Rather these subjects were focusing on the non-critical attribute for a rectangle of having two long sides and two short sides which was not always mentioned in their definitions. 35% did not identify a rotated rhombus as being a parallelogram.

Many students had incorrect definitions of various terms. For example, an isosceles triangle has exactly two congruent sides and a rectangle has two long sides and two short sides. Other students were unsure of the definitions of mathematical terms such as isosceles, congruent and hypotenuse. For terms such as similar, they attempted to use the English language definition of the word to apply to the mathematical problem. For example, if they had not been specifically taught the mathematical meaning of the term "similar", students used definitions such as "It's like the same, but there might be a very slight little difference.", "Two figures look a little, sorta like each other.", or "Congruent means exactly. Similar, you know, means maybe like this much off and stuff." Most of these gifted students attempted to deduce the definitions of terms they were unsure of from the context of the questions. They would then base their answers upon their conjectured meanings, no matter how conceptually inadequate they might be. Generally, the subjects were consistent, given the definition they were basing their thinking on, and often quite sophisticated in their reasoning. When they were faced with a contradiction or inconsistency, they would generally fault their definition. Some students would simply give up at this point, while others would attempt to change or refine their definitions. They enhanced their definitions as they detected additional information in the context of subsequent questions as well. In dealing with terms they had an existing schema for, such as the English language definition of "similar", subjects had a tendency to persist in their definitions. Apparent inconsistencies were often ignored.

Many subjects exhibited a lack of knowledge of properties, which should be mastered at Level 1. For example, 47% of the 43 subjects said that a right triangle doesn't always have a largest angle. The other 53% (23 subjects) said a right triangle always has a largest angle and 18 of the 23 identified it as the right angle or as 90 degrees. When asked if a right triangle has a longest side, 53% of the 43 subjects answered no. Of the remaining 20 students (47%) who said a right triangle does have a longest side, only 5 could name that side as the hypotenuse.

An identical 47% correctly answered the Level 4 question:

Suppose you have proved statements I and II.
I. If p, then q.
II. If s, then not q.
Which statement follows from statements I and II?
A) If p, then s.
B) If not p, then not q.
C) If p or q, then s.
D) If s, then not p.
E) If not s, then p.

DISCUSSION

Only 7% of these gifted subjects were classified as having attained van Hiele level 2 as measured by the CDASSGP test. These findings are consistent with those of Fuys, Geddes, and Tischler (1985) with sixth and ninth graders. Analysis of the clinical interviews confirmed Mayberry's rejection of the hypothesis that an individual demonstrates the same level of thinking in all areas of geometry included in the school program (1981). Performance was hampered in many areas by subjects' lack of definitions, incorrect definitions and misconceptions. In many cases where the subject could give a reasonable definition if a figure such as a rectangle, they still depended on the shape of the figure as a whole to identify the figure rather than its specific features as enumerated in the definition they had given. Even when they utilized specific features of a figure for identification, they frequently focused on non-critical attributes.

The reasoning ability of the subjects was far beyond what may have been anticipated, given their lack of knowledge of basic definitions and concepts. In many cases, the students would build valid logic structures based upon their conjectured definitions. This type of thinking is indicative of Level 2, but has been accomplished without knowledge of specific definitions or geometric content. Deduction is meaningful to most of the subjects. In fact, many of them (47%) could manipulate symbols without referents according to the laws of formal logic. However, they have not been exposed to the "rules of the game" and so do not know how to construct an acceptable proof. It should be noted that deductive reasoning is a skill which can be developed outside the context of geometry as it apparently has with many of these subjects.

Generally, these students were capable of handling inclusion relationships if they had suitable definitions of the elements involved, a characteristic of Piaget's

Stage 3 as well as van Hiele's Level 2. But an equilateral triangle can not be an isosceles triangle if you think that an isosceles triangle has exactly two sides.

While the though patterns of these gifted subjects do not seem to be described well by the van Hiele theory of geometric understanding, they do need Level 1 and Level 2 experiences in order to provide a foundation for their reasoning, so that they do not have to deduct the meaning of the terms they encounter and the relationships. Provided with this additional background, gifted junior high students should be capable of a proof oriented geometry course.

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THE UNDERSTANDING OF TRANSFORMATION GEOMETRY CONCEPTS OF SECONDARY SCHOOL STUDENTS IN SINGAPORE

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This paper reports a study which investigated the van Hiele-like levels of understanding of transformation geometry concepts of secondary school students in Singapore. Results show a possible hierarchy of the levels where most students were at the lower two levels. The paper presents a description of the behavior of the students in transformation geometrical tasks.

Geometric transformations became a part of secondary school mathematics curriculum in Singapore in the 1970's. The hope was that children would get insights into the mathematical structure and the underlying unity of mathematics through learning of geometric transformations. In England, Kuchemann (1980) professed that these aims of transformational geometry were as inaccessible to many children as was the deductive geometry.

van Hiele (1958) proposed a five level sequential theory. Many studied these levels and showed them to be useful in explaining geometric understanding. See Soon (1989) for references. The question now is whether the model is equally applicable to the learning of transformation geometry. Hoffer (1983) proposed and applied the van Hiele model for transformation geometry. The goal of this research project was to investigate whether the model could help to explain the understanding of transformation geometric concepts (in particular, reflection, rotation, translation and enlargement) of Singapore students. In Singapore, students normally received instruction on transformation using a coordinate and matrix approach in a traditional classroom setting.

METHOD

A van Hiele-like level characterization for transformation geometry followed from interpreting Hoffer's proposal and relevant literature relating to the van Hiele model. The researcher developed questions fitting the descriptions of the levels. The list of questions and level of characterization, critiqued by a panel of mathematics educators suggested revisions and validation for the first four levels. A criterion level was set for each concept at each level.

Twenty students selected from a secondary school containing representatives of the two streams in Singapore tried the validated test items in one-on-one interviews. The students, ages 15-16, were in the fourth year in the secondary school. Each student went through two sessions of in-depth interviews of about one and half hour each with video or audio taping of the interviews. During the interview the researcher probed for understanding and thinking. In the analysis, two persons independently assigned levels based on students' responses. The researcher followed the procedures used by Mayberry (1981) to analyze the data for patterns of behaviors for possible level hierarchy using a Guttman Scalogram.

RESULTS

Analysis of the response patterns of the students seemed to support a possible hierarchy of the levels. It revealed that most of the students were at Level 1 and Basic Level. The percentage of responses at each level of thinking was: Basic, 42.5 %; Level 1, 36.25 %; Level 2, 6.25 %; Level 3, 12.5 %.

Students showed limited use of precise language. In describing each transformation they frequently used their fingers to show movement. For rotation and reflection, students commonly used the word "move." Students described the image due to reflection as "opposite," or "left side becomes right side" to convey reverse orientation. The subjects seldomly used the words "congruence" and "similarity," concepts generated as a result of transformations. They employed descriptions such as "fit each other," "map onto each other," "equal," "equal angle but sides enlarged two times." The students used the word "mapping" quite loosely and interchangeably with the ideas of congruence and similarity.

Their success in the transformation tasks seemed to be in the order reflection, rotation, translation and enlargement. The students often recognized and viewed each transformation as a whole for motion. The students did not spontaneously give the specifics -- center and angle of rotation, center and scale factor for enlargement, line of reflection, and translation vector -- in describing each transformation. The students gave these responses only if elicited by the researcher. The location and size of these specifics seemed to affect their performances.

Students often saw rotation as a turning effect. In one of the items (see Figure 1a) students were to identify a single transformation and locate the center of rotation. Figure 1b showed one of the solutions presented by one of the students. She viewed it as a translation followed by a rotation with the foot of the figure as the center. She was unable to give a single transformation. In the same task, most students could nearly locate the center of rotation by viewing but had difficulty doing it accurately.

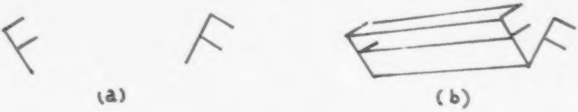
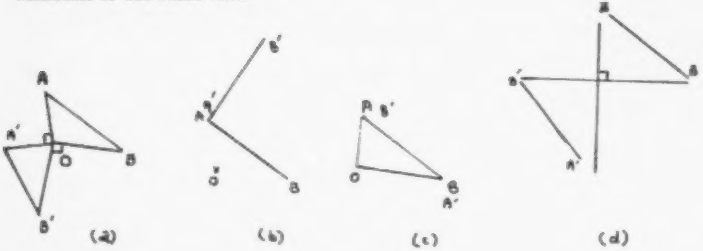

(a) (b)

Figure 1. A Test Item and Solution of a Student

In another rotation task, students were to draw the image of a line segment AB rotated 90 degrees counterclockwise about a point O external to AB. The same student, disregarding the direction of turning, used the end points A and B, rotated 90 degrees, and located the images A' and B' as shown in Figure 2a. A second student focused on A as center and rotated AB as indicated in Figure 2b. This latter student completely disregard the point O. Figures 2c and 2d show two other solutions to the same task.



(a) (b) (c) (d)

Figure 2. Solutions of Students to a Test Item

Another item on rotation seemed to reveal that students have problems with visualization. In this item, students were to give the coordinates of the image C' of C after a clockwise rotation of ABC, 90 degrees about the origin. They were given the coordinates of A, B, C and B' and the figure showing the respective positions with the coordinate axes as shown in Figure 3. Students were to estimate the location of C'. C1, C2, and C3 show some of the estimations presented by the students. None estimated the location correctly, however, most were able to complete the task accurately and correctly using compasses and ruler.

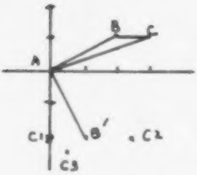


Figure 3. Estimation of Location of C'

The scale factor in enlargement confused the students. To many students, a positive scale factor produced a larger image and a negative factor produced a diminished image. The responses of the students to the properties of transformations revealed that they perceived enlargement as one which always changed the size of figures.

For translation, many students viewed each movement as composed of two components -- horizontal and vertical displacements. The students were introduced to translation as a column matrix. However, they had difficulty interpreting the translation vector geometrically.

Although their instruction uses a coordinate and matrix approach to transformation, students had difficulties in relating a given matrix to a visual representation. However, most were able to tell certain transformations associated with matrices taught by their teachers. The students memorized these matrices. Students were very proficient with operations on matrices.

At Level 2 as proposed by Hoffer (1983) for transformation geometry, the students are to relate the properties of the transformations, for example, the composition of two reflections being equal to a rotation or translation. This was not in the curriculum, so they were unable to use these interrelated properties in solving the items.

At Level 3, where items required them to do proofs, most students gave particular examples as solutions. To prove a given statement, most students investigated using instances by way of drawing. Later they showed the particular example as proof. In presenting proofs of congruency or similarity of two geometric figures, students often used a visual approach by initially comparing corresponding sides and angles. They then proceeded to identify the correct transformation ignoring the specifics in describing the particular transformation. Quite frequently, they gave incorrect specifics of the transformation especially for enlargement. Some of the reasons for proof of congruency or similarity were "no changes to angle or sides," and "mapping onto each other."

During the interviews, the researcher questioned the students on their solutions. Quite often, they referred to the teacher and the text as reasons for their solutions. In addition, they would say "I can't remember" when a certain procedure, which they knew would help them to solve the problem, was forgotten.

One student who reached Level 3 for three of the concepts and Level 2 for enlargement could give reasonable explanations to all her solutions, and even conjectured at Level 2 for the interrelated properties of the transformations.

DISCUSSION

The van Hiele-like levels seemed to explain the behavior of the students as seen in the study. They viewed each transformation as a whole in motion before attending to properties and specifics of each transformation. The students in this study had transformations introduced through coordinates and matrices perhaps contributing to their deficiencies in pictorial representation and behavior for the rotation items mentioned above. However, the intuitiveness of transformation enabled them to answer the test items at the Basic Level. Their lack of precise vocabularies in describing transformation points to the language that van Hiele (1958) placed so much emphasis on at each level. Their reference to authority for confirmation of solutions reflected the traditional school mathematics environment. A more dynamic problem-solving approach to instruction is being advocated now where students work in groups to explore, observe, discover, predict, conjecture, and communicate. This would allow them to be in control of their own learning. In addition, the students should be exposed to more real-life applications of transformation to develop meaningful learning.

The results and findings are limited to the twenty students, all female, under this study. Future study should include a broader range of students to look into the higher levels of thinking.

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MULTIPLICATIVE STRUCTURES

PROPORTIONAL REASONING IN YOUNG ADOLESCENTS: AN ANALYSIS OF STRATEGIES

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The present study employs a qualitative proportional reasoning task incorporating increase, decrease, or stays the same transformations on the numerator or denominator in a non-numeric setting in order to determine the influence of instruction, problem type, problem format, gender, and individual differences in strategy use. Prototypes of subjects' strategies and percent correct scores from responses are described.

One ubiquitous content domain for the study of the cognitive development of individuals is proportional reasoning--"a term that denotes reasoning in a system of two variables between which there exists a linear functional relationship" (Karplus, Pulos, & Stage, 1983, p. 219). Numerous problem types and contexts have been employed to investigate strategies, errors, and complexities of proportional reasoning abilities. For instance, problem context, discrete vs. continuous measures, problem format, numeric vs. qualitative problem type, subject characteristics, and gender constitute significant factors that may influence individual and group proportional reasoning problem solving, some of which have been studied.

Strategy development and use are described by many researchers (e.g., Noelting, 1980). Noelting (1980) described individual differences in strategies using a numeric lemonade mixture problem context. Pulos, Karplus, and Stage (1981) examined problem content (discrete vs. continuous objects), numerical structure, age, and gender and found that numerical structure effected mean differences in subjects where equal-integer problems resulted in higher performance compared to unequal-non-integer problems. Also, Heller, Ahlgren, Behr, and Lesh (in press) found that familiar rate types in the context of qualitative numeric proportional reasoning problems were easier for students to solve than less familiar rate types. Students have difficulty with more/more and less/less qualitative proportional reasoning problems highlighting the effects of rate type on student performance (Heller et al., in press).

The present study employs a qualitative proportional reasoning task incorporating the same types of combinations of transformations used by Noelting

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(1980) and Heller et al., (in press)--that is, increase, decrease, and stays the same--yet in a non-numeric setting in order to determine the influences of instruction, problem type (discrete vs. continuous contexts), problem format, gender, and individual differences on strategies used in problem solution by early adolescents.

METHOD

Subjects

Eleven seventh-grade children from Minneapolis, Minnesota (4 females, 7 males) and ten seventh-grade children from DeKalb, Illinois (4 females, 6 males) were selected as participants in a teaching experiment. The teaching experiment was conducted during the spring semester of 1987 over a period of about 17 weeks. The instruction incorporated a heavy emphasis on active experiential learning based on various tasks incorporating proportional reasoning. High, middle, and low mathematical ability levels equated the Minnesota and DeKalb groups. Two subjects did not complete the pretest, posttest, or items from the first three interviews.

Procedure

Qualitative proportional reasoning problems were selected for investigation in the present study. After drawing a box-fraction representation ($\frac{\square}{\square}$), the experimenter stated, "Here is a fraction. It has a top number and a bottom number." Then questions about differing transformations were asked of the subjects (e.g., "What happens to the value of the fraction if the top number increases and the bottom number decreases?").

Six numerator-denominator transformations defined by combinations of increase (I), decrease (D), and stays the same (S)-- I/D, D/I, D/D, S/I, I/S --were presented.

Problems were presented in a fraction, ratio, or chart format. During interviews I and II, the experimenter referred to the drawn representation as a fraction with a numerator (top box) and denominator (bottom box). However, in interview IV, the emphasis changed from fraction to ratio, and one box represented some number of rectangles (i.e., the numerator) and the other box represented some number of squares (i.e., the denominator). Some number of cans of red and white paint corresponded to boxes designated in the second part of interview IV. In each part of interview IV, four problems were presented in ratio format and four in chart format.

Strategy Description

Following data collection by video recording, subjects' verbal responses were transcribed and analyzed in order to determine the type and frequency of strategies used to solve the tasks. Eight strategy categories accounted for all of the responses: namely, qualitative transformation strategy (QLTS), quantitative transformation strategy (QNTS), reference point strategy (RPS), whole number dominance (WND), reciprocal rate strategy (RRS), manipulative strategy (MS), and unit rate method (URM). Other categories were used to describe undetermined responses (OTR=other), no reasons given (NRG), and missing data (MD). Some strategy categories were further subdivided to discriminate among substrategies within categories. Detailed descriptions of general categories and subcategories are subsequently discussed.

Qualitative transformation strategy (QLTS). Qualitative transformation strategies consist of intuitive ideas about the change or no-change in the size of fractions and ratios under various transformations in the absence of any reasoning based on numerical manipulations or transformations. This general reasoning strategy about transformations of fractions is further divided into 8 substages describing types of qualitative reasoning subjects used. These substages, along with prototypical responses from subjects are subsequently described. The responses were given to questions like: What happens to the fraction (ratio) if the top number (increases, decreases, or stays the same) and the bottom number (increases, decreases, or stays the same)?

Descriptive (DES). The subject describes a situation, provides a rationale, or elaborated on an answer.

> Interview I (Inc, Inc) Ann: "The bigger the numbers are the smaller the fraction...more you cut, the less it is."

Transform numbers--fraction increases. The subject states or implies that, because both of the numbers change in the same direction, the fraction increases.

> Interview II (Dec, Dec) Shannon: "5/6--4/3... . No, 5/6--4/5. Increase...Yes any time both numbers decrease then the fraction will increase."

Transform numbers--fraction decreases. The subject states or implies that, because both of the numbers change in the same direction, the fraction decreases.

Transform numbers--fraction stays the same. The subject states or implies that, because both of the numbers change in the same direction, the fraction stays the same.

More than strategy. Subject states that if you transform one dimension (e.g., white paint, the numerator, or squares) more than another (e.g., red paint, the denominator, or rectangles) then a certain outcome will occur.

> Interview IV (Dec, Dec) Ann: "stays the same because decrease both...(I: Always?) no if you decrease one more than the other...if you decrease white more than red then darker...if you decrease the same then stays the same...if decrease red more than white then lighter."

Transform--change. Subject states that if a dimension increases (or decreases) more than another dimension then the fraction changes in value.

Action-outcome, Action-outcome. Subject states one outcome based on a transformation, then states a second outcome (usually similar but to a greater degree of the first outcome) based on another transformation.

> Interview IV (Dec, Inc) Jon: "3/4...if decrease the numerator then gets smaller but if increase denominator then gets even smaller."

You don't know-rule based (YDK-R). This rationale is based on a mental rule stated by the subject (learned or spontaneously produced by the subject). No examples are given by the child to substantiate the rule.

> Interview II (Dec, Dec) Dave: "Didn't change. They don't tell you how many more cans of red paint and they don't tell you how many more cans of white paint...you don't know if it changed."

Quantitative transformation strategy (QNTS). A quantitative transformation strategy describes reasoning about transformations of fractions and ratios that is based on numerical representations or examples. The subject either provides numerical examples to substantiate an answer or uses numbers in the process of figuring out transformation problems. Three subcategories are listed.

Reduce then compare (R-C). The subject first reduces one or more fractions then makes a comparison.

You don't know--understand (YDK-UND). The subject gives examples to support his/her statement "you don't know". Subjects who answer "It depends" to the question "What happens to the value of the fraction?" and follow this answer with examples to substantiate the response fall into this category also.

> Interview IV (Dec, Dec) Sharon: "It depends. If 2R 4W--1R 2W (R=cans of red paint, W=cans of white paint) same. If 2R 4W--1R 3W, smaller. If 2R 4W-- 1R 1W, bigger."

Number of times strategy. Describes the change in fractions after transformations by considering the number of times that different dimensions (white paint, rectangles) decrease or increase.

> Interview IV (Dec, Dec) Shannon: "It depends how much you decrease it... If decrease the same number of times (3 times) then same... If decrease R 4 times and W 2 times--redder. If decrease W more than R than lighter."

Reference point strategy (RPS). The subject compares fractions to a whole or to one-half.

Compares to a whole-additive (CTW-A). The subject makes an additive comparison to the whole.

Compares to a whole-multiplicative (CTW-M). The subject makes a multiplicative comparison to the whole.

Whole number dominance (WND). Subject compares or discusses whole numbers rather than comparing fractions.

> Interview II (Dec, Dec). Jody: "6/5--5/4. Decreases. Because 6 is greater than 5 and 5 is greater than 4."

Reciprocal rate strategy (RRS). Subject compares reciprocal rates rather than rates asked by interviewer.

Manipulative strategy (MS). Subject uses physical representation to determine response to transformation.

> Interview IV (Dec, Dec). Carrie: Draws a pie for 4/5 and shades in 4 pieces. Draws a pie for 3/4 and shades in 3 pieces. "4/5, 3/4, 2/3, 1/2...It's getting bigger. 4/5 is less than 3/4."

Unit rate method (URM). Subject determines a unit rate in order to determine an answer.

RESULTS AND DISCUSSION

Trends for mean percent scores and frequencies of strategy use indicate the existence of differences in factors related to proportional reasoning. Due to the small subject sample, the data were not statistically analyzed, but highlight important variables.

Strategy Data

1. Protocol analyses revealed distinct, differential strategy use between subjects. Totals for existent strategies indicate that the qualitative transformation strategies were used to a large extent by subjects (175), compared to moderate use of the quantitative transformation strategies, reference point strategy, additive strategy, and

manipulative strategy (25, 21, 21, and 27, respectively). Finally, the unit rate method and whole number dominance strategies were used very infrequently.

2. Subjects performed best on the increase/same problem type--95% overall. The following problem types were next in rank: same/increase (84%), increase/decrease (81%), decrease/increase (79%), decrease/decrease (58%), and increase/increase (55%). Although performance improved for many problem types from interview one to interview four, note the low performance for problems where the numerator and denominator change in the same direction.

3. Subject's overall mean percent correct scores were higher for continuous problems (red and white paint) compared to discrete problems (rectangles and squares)-- means are 65.87% and 81.50%, respectively.

4. Subjects obtained higher percent correct scores on problems in chart format compared to fraction format (means are 70% and 62%, respectively for interview four part one and 85.5% and 77.5%, respectively for interview four part two).

5. Mean percent scores for subject ability levels indicate that performance is in the expected direction; high ability mean performance was 83.7%, middle ability mean performance was 62.8%, and low ability mean performance was 56.2%.

Results suggest that individuals differ in strategies used to solve qualitative proportional reasoning problems and several factors influence problem solution.

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YOUNG CHILDREN'S THEOREMS-IN-ACTION ON MULTIPLICATIVE WORD PROBLEMS

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This post hoc analysis reports the use of Vergnaud's measure space diagram of multiplicative structures to characterize children's theorems-in-action on multiplication and division word problems. The theorems-in-action also were identified as scalar or function theorems. Vergnaud's diagram works well to depict differences in children's theorems-in-action. Children used only scalar theorems on multiplication and measurement division problems, but both scalar and function theorems on partitive division problems.

In his analysis and description of how children approach multiplication and division word problems, Vergnaud (1988) identified two basic kinds of "theorems-in-action" (mathematical relationships attended to by students when solving a problem): a scalar approach and a function approach. Using Vergnaud's (1988) 4-quantity, 2-measure space description of multiplicative structure, the scalar method involves multiplying within a measure with no change in the kind of quantity (see Figure 1, e.g., a objects x b = c objects), whereas the function method involves multiplying across measures (see Figure 2, e.g., b groups x a = c objects).

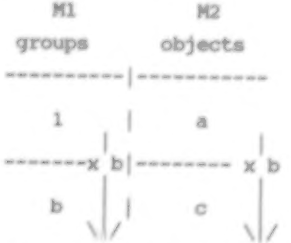

Figure 1. Scalar method of solving.

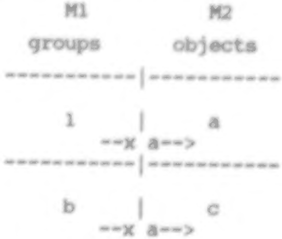

Figure 2. Function method.

This study was a post hoc analysis of a previous study (Kouba, 1989) of children's solution strategies for multiplication and division word problems. One purpose was to determine whether the data supported Vergnaud's (1988) conclusion that children used scalar theorems-in-action when solving simple multiplication problems. A second purpose of this study was to determine whether children used scalar or function theorems-in-action to solve simple measurement division problems (i.e., find the number of groups) and simple partitive division problems (i.e., find the number of elements in a group). A third purpose was to explore how Vergnaud's 4-quantity, 2-measure space diagrams for multiplicative structures may best be used as a means for describing children's solution strategies in a step-by-step linear fashion.

The children's theorems-in-action for multiplication and division problems were analyzed and categorized based on which of the four quantities in Vergnaud's diagram (1, a, b, and c) were represented physically. The theorems-in-action also were characterized based on the order that children mathematically processed the quantities and the relationships among the quantities depicted in Vergnaud's diagrams. Then these analyses were used to classify the theorems-in-action as scalar theorems or function theorems, if possible.

Subjects. The subjects were 43 first-grade, 35 second-grade and 50 third-grade children.

Procedures. In an individual interview, each child solved 2 multiplication, 2 measurement division, and 2 partitive division word problems. The problems involved disjoint equivalent sets of whole numbers. The children had physical materials available to use while solving. The children's responses were coded and tape-recorded.

RESULTS

Multiplication. There were five types of representations with physical objects used for multiplication problems: 1) The quantities a, b, and c were represented with physical objects. The child set out b containers, filled each container with a objects, and calculated the total number of objects (c) in varying ways (e.g., counted one-by-one, counted on one-by-one from a, counted by multiples of a, "counted" by repeatedly adding a, a combination of multiple counting and counting on, or a combination of repeated addition and counting on.) 2) The quantities a and c were represented with physical objects -- similar to Type 1 except no containers were used. 3) The quantity a was represented with physical objects. The child made one group of a objects and "counted" the group b times in varying ways. 4) The quantity b was represented. The child "counted" b groups of a, either by use of repeated addition or counting by multiples of a. One object was set out for each group of a that was "counted." 5) No quantities were represented using physical objects. Calculations were done using counting by multiples, repeated addition, derived number facts, or recalled number facts.

All five types of representations were used at each grade level; however, Grade 1 children used primarily Types 1 and 2, Grade 2 children used all five types uniformly, and Grade 3 children used primarily Type 5.

All of the theorems-in-action used with representa- tion Types 1-4 were scalar in nature because the children's actions were limited to creating groups of a objects b times. The theorems-in-action differed only in the level of abstraction used within the representation of quantities or within the kind of counting or calculating that was done. Children who used Type 5 representations other than derived number facts or recalled number facts also used the number of groups as a scalar operator. The five children who used derived number facts used scalar theorems-in-action as well. For example, for a problem involving calculating the total number of nuts under a tree if there were 4 trees and 6 nuts under each tree, children reported thinking, "6 plus 6 is 12, 6 plus 6 is 12, and 12 plus 12 is 24." It was impossible to judge whether the recalled number facts used by children were scalar or function in nature, because during the interview no questions were asked to elicit whether the children were thinking b times a objects or a times b groups.

Measurement Division. Theorems-in-actions used to solve measurement division problems were classified on two dimensions. One dimension was on type of representation and resulted in seven types. The other dimension was on whether children used an "exhaustive take-away" theorem-in-action, a "building-up" theorem-in-action, or a recalled number fact.

The seven types of representations were: 1) The quantities a, b, and c were represented with physical materials. For the exhaustive theorems-in-action the child counted out c objects, set out one container and filled it with a objects taken from c, repeating this until $c = 0$. For the building-up theorems-in-action the child set out a container, filled it with a objects, repeated the action of filling containers while keeping a running count of the total number of objects that had been placed in containers, and stopped when the count reached c. 2) The quantities a, c, and 1 container were represented with physical materials. Only an exhaustive theorem-in-action was used -- similar to Type 1 except that a container was used only with the first group. 3) The quantities a and c were represented with physical materials. These methods were similar to Type 1 methods, except that no containers were used, objects were just put in groups of a. 4) The quantities a and b were represented physically. Only a building-up theorem-in-action was used. The child made a group of a objects and counted it repeatedly (in a variety of ways), keeping a running total until c was reached and setting aside a new object each time a was counted. 5) The quantity a was represented with physical objects. Only a building-up theorem-in-action was used. The child made a group of a objects and counted it repeatedly (in a variety of ways) until a total count of c was reached. 6) The quantity b was represented with physical objects. Only a building-up method was used. The child counted by groups of a (in a variety of ways) until the total count reached c. An object was set aside for each group of a that was counted. 7) No quantities were represented physically. Both exhaustive and building-up methods were used. The exhaustive method was repeated subtraction. The building-up methods included repeated addition, systems similar to Type 5 and Type 6 but with a and b represented internally, and derived number facts. Children also used recalled number facts, which were neither exhaustive nor building-up theorems-in-action.

Grade 1 children used only exhaustive take-away theorems-in-action of Types 1, 2 and 3, primarily 1 and 3 (one child did use one repeated addition with no physical objects). Grade 2 children used Types 1, 3, 6, and 7, primarily Type 3. About 2/3 of Grade 2 children used exhaustive take-away theorems and about 1/3 used building-up theorems. Grade 3 children used all seven types, but about 60% used derived or recalled number facts.

All of the theorems-in-action of Types 1-6 were scalar in nature because they involved using the number of groups as a scalar operator. Likewise, all of the Type 7 theorems except recalled number facts were scalar. Recalled number facts could not be classified.

Partitive Division. Theorems-in-action used for partitive division problems were classified by type of physical representation and by order of processing steps (exhaustive take-away or building up). Theorems-in-action for partitive problems also were classified as "dealing-out" theorems or "guess at a" theorems. The dealing-out theorems consisted of representing or visualizing b groups and dealing c total objects to those groups one-by-one (often called "sharing fairly"). Guess-at-a

theorems consisted of guessing at how many objects would be in a group and then performing theorems similar to those for measurement division problems.

There were six types of representations with physical objects: 1) The quantities a, b, and c were represented with objects -- (a) exhaustive guess-at-a theorems similar to Type 1 measurement division theorems, or (b) exhaustive and building-up dealing-out theorems. 2) The quantities a and c were represented --(a) exhaustive and building-up guess-at-a theorems similar to measurement division Type 3, (b) exhaustive dealing-out, or (c) exhaustive grouping theorem for which the child counted out c objects, divided them into b groups that were unequal in number of elements, and then redistributed elements from the groups until the groups were equal in number (a objects in each group). 3) The quantities b and c were represented -- the child formed groups of b objects while keeping a running count of the total, stopped when the count reached c, counted the number of groups formed (a) and said that each group had a objects. 4) The quantity a was represented -- The child counted by multiples of b until c was reached, keeping track with objects of how many multiples were spoken (a). 5) The quantity b was represented -- building-up, guess-at-a theorem similar to measurement division Type 6. 6) No quantities were represented with physical objects. Both exhaustive and building-up guess-at-a theorems were used, as well as derived and recalled number facts.

No Grade 1 children used building-up theorems; most Grade 1 and Grade 2 children used representation Types 1 and 2. Grade 3 children used primarily recalled number facts.

The guess-at-a theorems-in-action were scalar in nature, whereas the dealing-out theorems were function in nature. Representation Types 3 and 4 also were function in nature, but appeared more abstract than dealing-out theorems and were the only theorems where the b quantity took on the role of being elements of a group in the same way that the a quantity is usually treated. All of the derived number facts were scalar in nature, although one might expect children to have used a function theorem. For example, for a problem involving 20 objects that were to be distributed equally into 4 groups, rather than using the "4" given in the problem and saying "4 + 4 is 8, 8 + 8 is 16, and 4 more is 20, thus the answer is 5," children estimated the number of objects in a group and said "5 plus 5 is 10, then another 5 and 5 is 20; so 5 works."

Both scalar and function theorems-in-action were used at all three grade levels. However, most of the children used one or the other method, but not both. Only 4 of the thirty children who solved both partitive division problems correctly used a scalar theorem on one problem and a function theorem on the other. Therefore, most children did not demonstrate that they could use both theorems interchangeably.

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STUDENT USE OF RATIONAL NUMBER REASONING IN AREA COMPARISON TASKS

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The purpose of this study was to determine whether 4th-, 6th-, and 8th-grade students would recognize the need for and apply their knowledge of fractions i.e., use Rational Number Reasoning to solve comparison of area tasks. Thirty six students participated in individual task-based interviews. Their video taped responses were analyzed for the types of strategies they used to solve the tasks. A very low percentage of the responses revealed the use of Rational Number Reasoning.

THEORETICAL FRAMEWORK

According to the results from research on students' concepts of fractions and the National Assessments of Educational Progress most students do not understand fraction concepts and cannot apply them in problem solving situations (Post, Behr, & Lesh, 1986; Behr, Wachsmuth, & Post, 1985; Carpenter, Corbitt, Kepner, Jr., Lindquist, & Reys, 1981). Researchers have also found that area concepts are not well-developed in younger students and, in fact, continue to present problems even into adolescence (Piaget, Inhelder, & Szeminska, 1948/1960; Hirstein, Lamb, & Osborne, 1978). However, the area model, in the form of shaded parts of geometric regions, is the one most commonly used in elementary school for introductory work with fractions.

Recent revisions in state and national curriculum standards make reasoning and problem solving the focus of the mathematics program. The ability to reason is directly affected by formal education (Luria, 1976), and the individual's ability to reason develops over time (Inhelder & Piaget, 1958, 1964). Little is known about the ability of students to reason with the fraction concepts they are taught in school when they are not specifically told to use fractions to solve a problem (Kraus, 1977).

OBJECTIVES

It is the purpose of this study to determine whether students use rational number reasoning to solve comparison of area tasks and whether the tendency to use such reasoning increases with grade level. Rational number reasoning (RNR) is defined as the ability to perceive the logical mathematical structure of a problem which can be solved by the application of rational number knowledge. Students' behavioral and verbal responses to area comparison tasks in individual clinical interviews comprise the data used to answer the following questions:

1. To what extent do 4th-, 6th-, and 8th-grade students use RNR when comparing the partitions of two areas?
2. Does the frequency of RNR responses increase when fraction symbols are introduced into the tasks?

METHODOLOGY AND DATA SOURCE

Sample

Twelve students each from the fourth, sixth, and eighth grades were randomly selected for the study from an elementary and a junior high school in Tucson, Arizona. The populations attending the schools are ethnically diverse and range in socio-economic levels from lower to upper-middle.

Tasks

Six boys and six girls at each grade level were individually presented with 21 area comparison tasks during an interview which lasted an hour or less. The tasks consisted of comparing the shaded partitions of two rectangular regions. In 19 of the 21 tasks the regions were partitioned in different directions (horizontally or vertically), therefore creating partitions which were not congruent. Each of these tasks varied by type of fraction (unit or proper multiple) or by size of unit (same or different). In the remaining two tasks the unit sizes were the same, the same fractions were used (7/16), and partitions were congruent, but on one area the shaded partitions were scattered and on the other area they were clustered together. This task was an area conservation task to test whether the arrangement of the shaded areas influenced the students' reasoning about the task.

The first 13 tasks had no fraction symbols or terms associated with them. As part of the last 8 tasks the students were asked to tell and write how much of each area was shaded. The students wrote fractions for each model on Post-its which were attached to the models. The students were then asked to compare the shaded areas again in order to see if the fraction symbols triggered their use of Rational Number Reasoning.

All of the conditions were present in each task for the studentsto deduce the answer by using Rational Number Reasoning. After the students answered each of the task questions they were asked why they answered as they did. Their responses to the question "Why?" along with the behaviors they exhibited gave insight into the kind of strategies they used to solve the tasks.

The interviews were video taped, and the students' explanations and the actions they performed during the tasks comprised the data for the study.

RESULTS

The total number of student responses was 756. These responses were analyzed for the types of reasoning and comparison strategies the students used to solve the tasks.

The students' explanations and behaviors indicated that they used different types of strategies to solve the tasks. Two broad groups of strategies emerged, those in which students focused on directly comparing properties of the models, and those in which the students noted and used the part-whole relationships inherent in the tasks.

Five categories emerged in which students directly compared properties of the models. Direct comparison in this case represents the thinking that the students verbalized, gestured, or actually demonstrated by moving the models. Responses in these categories made up 68% of the total number of student responses. The categories are as follows:

1) Only Area of Part(s)--Direct comparison of shaded or unshaded part(s)
2) Only Length of One Dimension of Parts--Direct comparison of either the widths or lengths of the shaded or unshaded parts.
3) Only Number of Parts--Direct comparison of number of shaded or unshaded parts.
4) Combination of Properties--Direct comparison of more than one of the properties, and
5) Other--Responses too vague or which did not fit any of the identified categories.

Four categories of reasoning were identified in which students formed or partially formed part-whole relationships. Responses in these categories made up 32% of the total. The categories are as follows:

1) Rational Number Reasoning--Fraction terms associated with and used to compare shaded areas, unit sizes compared, and conjunction used to relate parts to wholes.
2) Partial Rational Number Reasoning--Fraction terms associated with and used to compare shaded areas, one or more of the other RNR conditions were omitted.

3) Area Part-whole--Number of parts in whole compared, number of parts shaded compared, size of unit compared.

4) Area Partial Part-whole--One of the conditions was omitted from those in Area Part-Whole category.

Over all of the tasks percentages of RNR responses by the students in this study were low. The percentages of RNR responses by students at the three grade levels are as follows: fourth graders 2%, sixth graders 3%, and eighth graders 4%.

In order to answer the second question of this investigation the percentage of RNR responses is tabulated by tasks without and with symbols introduced. As can be seen in the table below, the percentage of responses at each grade did not vary a great deal for the tasks without symbols. After the symbols were introduced the percentage of RNR responses at each grade level increased slightly, with the 8th graders showing the highest percentage of increase, 8%, while the 4th and 6th graders only increased by 2%.

Percent of Rational Number Reasoning Responses Without and With Fraction Symbols at Each Grade Level

	% RNR	
Grade	Without[a]	With[b]
4	1	3
6	2	4
8	1	9

[a] Percentage of total number of responses (468) to tasks presented without association of fraction symbols.

[b] Percentage of total number of responses (288) to tasks presented with the association of fraction symbols.

CONCLUSIONS

Even though area models are prevalent in fraction instruction throughout the elementary school grades, most of the students did not use fraction knowledge to compare the shaded parts of the rectangles. The introduction of fractions into the tasks did seem to trigger the use of RNR much more frequently by the eighth graders, but on their own they did not recognize that the structure of the task called for the use of their fraction knowledge.

This study provides insight into the thinking of elementary and junior high school students as they solve comparison of area tasks. The study also provides the opportunity to identify the kinds of task conditions to which students attend when their thinking is not directed to particular conditions. It is important to gather information in this area in order to understand the kind of reasoning students use when faced with novel problems.

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Multiplicative Word Problems
- Recent Developments

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ABSTRACT

Five types of structure are identified in asymmetric (isomorphism of measure) problems requiring one operation of multiplication or division; these are Multiple Groups, Repeated Measure, Rate, Change of Size and Mixtures.

Performance on such questions is dependent on the context, the structural type and the types of number occupying certain roles in the problem. In multiplication problems, difficulty is strongly dependent on the type of number in the 'preferred multiplier' role.

In division problems, choice is dominated by the *numerical preference* for dividing by an integer and/or the smaller of the two numbers, particularly if the one is divisible by the other; decimal points are often ignored in determining this preference. The misconception that multiplying makes bigger and division smaller operates, but not in all structures.

In a recent experiment half the subjects were asked to give an estimate of the approximate answer, and half to choose the operation. Estimating the outcome was easier than choice of operation in division, and in multiplication by numbers less than 1. In multiplication by numbers substantially greater than 1, the reverse was true.

Introduction

A considerable body of research now exists on pupils' comprehension of problems embodying multiplication and division. This article extends the range of structures so far studied, to include Change of Size and Mixture problems, where the two quantities may be measured either in the same or in different units, and also compares results in two response modes, estimate and choice of operation.

Vergnaud (1988) has drawn an important distinction between *isomorphism of measures*, involving a correspondence between two quantities measured by numbers, and *product of measures*. In this article, we are concerned with the first of these; in such problems, multiplication is always asymmetric, multiplier and multiplicand playing distinct roles. A classification of such problems is shown in Table 1.

Table 1
A Classification of Asymmetric Multiplicative Situations

showing the preferred multiplier, as empirically determined

Structure	Multiplication	Multiplicand	Multiplier
Multiple groups	3 boxes contain 4 eggs each. How many eggs are there altogether?	4 eggs/box	3 boxes
Repeated measure	A gardener needs 3 pieces of string each 4.6 metres long. How much string should he buy?	4.6 metres/ piece	3 pieces
Rate	A man walked at an average speed of 4.6 miles per hour for 3.2 hours. How far did he walk?	4.6 miles/ hour	3.2 hours
Change of size (same units)	A photograph is enlarged by a factor of 4.6. If the height was originally 3.2 inches, how high is the enlarged photograph?	3.2 inches	scale factor 4.6
Change of size (different units)	A model boat is made to a scale of 4.6 metres to an inch. If the model is 3.2 inches long, how long is the boat?	4.6 metres/ inch	3.2 inches
Mixture (same units)	A painter makes a particular colour by using 4.6 times as much red as yellow. How much red should he use with 3.2 pints of yellow?	3.2 pints	scale factor 4.6

Mixture (different units)	4.6 lbs. of powder are to be mixed per gallon of water. How many lbs. should be mixed with 3.2 gallons?	4.6 lbs/ gallon	3.2 gallons

In the first two structures listed here, there is an integral number of repetitions of either a set of objects, or of a continuous quantity. The remaining structures all consist of a correspondence between two such quantities. Rate is the most general case of this. In Change of Size the correspondence is between versions of the same quantity, which may or may not be measured in different units. Map scale problems are an example in which the units are normally different. In Mixture problems two component quantities are physically brought together; they also may be measured in the same or different units.

The distinctions between an explictly stated rate, eg. 'How many miles per hour?' and 'How many miles in one hour?', or 'in each hour?' are also psychologically important and are reflected in substantial differences in difficulty.

In the first two structures, Multiple Groups and Repeated Measures, the multiplier is (by definition) the number of repetitions. In the later structures, the preferred multiplier is the quantity which plays the analogous role . Which of the two quantities this is, is not always clear *a priori* . The identifications in Table 1 are those determined empirically

In Table 1 the classes are represented by multiplications. There are also two types of division problem. These are *partition,* which in the first two structures means division of the total amount into a given number of parts, the result being the size of each part, and *quotition*, in which the size of each part is given and the number of them is required. We have extended these concepts into the later structures, the number of parts generalising to the multiplier.

Previous work in this field has studied the effects on recognition of the correct operation of different types of number, structure and context

(Bell, Fischbein & Greer, 1984). The results show strong numerical misconceptions, that multiplication makes bigger and division smaller (MMBDS), and that division must be of a larger number by a smaller one. These factors interact with the generally good perception of the size of the numbers and of the quantitative relations in a problem, to lead to erroneous choices of operation.

Incidence of numerical misconceptions

The numerical misconception MMBDS appears to operate not in all cases, but only in those where there is a natural comparison between size of operand and result. For example, it operates in '0.7 hours at 8 miles per hour, how many miles?', but not in '2.4 metres of cushion material cut into lengths of 0.48 metres. (Bell et al, 1984)

In Rate partition questions, (e.g. 0.75 hg cocoa cost 900 lire, how much does 1 hg cost?) MMBDS clearly operates. Rate quotition questions also attract multiplications, which suggests that MMBDS operates here, though not in pure quotitions. An example is: 'What capacity in litres has a 5.5 gallon fuel tank, if a litre is 0.22 gallons?' (Bell et al., 1984). We can see that, in this last case, there is a natural comparison between the capacity in gallons and the perceived greater litre-measure of the same capacity, which might produce the pull towards multiplication. However, rate quotition questions are susceptible to another factor which also tends to favour a choice of multplication. This is discussed below

Confusion of measures in rates

This factor came to light from observations of Rate Quotition questions requiring a smaller number to be divided by a larger. For example, 4.8 miles at 5.24 mph, how long?, attracts some reversal errors, as would be predicted, but even more multiplications. A striking difference appears between Rate Partition and Rate Quotition questions, in that when errors are made, the proportion which are reversals, rather than choices of multiplication, is high for partitions low for quotitions. These dominant errors both correspond to a confusion between the numerator and denominator roles in the rate, e.g. between miles per hour and hours per mile; this is stronger in the 'per' form than in the form 'miles in each hour'. We quote two pupils:"4 grams per penny and 4 pence per gram are the same, just swapped around"; "distance + time equals speed" "Why?" "km + hrs = km per hour" (Bell, Fischbein & Greer, 1984; Bell & Onslow, 1987).

Numerical preferences

Certain results are inexplicable by the above hypotheses and suggest an additional factor of numerical preference. For example, number combinations such as 7/23 and 2.4/0.48 attract more than the expected number of reversals (decimal points may be ignored). And a problem concerning the amount of meat for £2, if the price is £2.56 per pound, produced a fair number of subtractions; interviews suggested that the reasoning was that the answer would be less than a pound weight, but 2.56/2 is too much, where as 2.56 - 2 is an easy operation, and gives an answer of about the right size (Bell et al, 1984).

Structure

The relative difficulty of the different structures is unclear in most of the literature, since the supposedly harder structures are generally associated with the harder numbers. But there is some evidence that Rate questions are substantially harder than Repeated Measures given similar types of number. (Two quotitions 17.6/2.2 scored about 60%, two Rate Quotitions, in speed and mileage per gallon, 56/37.5, scored about 30% (Bell 1984). Within the structures of Table 1, there do not appear to be great differences between *contexts* (Bell, 1984) but this factor has not been systematically examined.

Evidence from problem writing tasks, to fit given calculations, is of strong tendencies to use the earlier structures of Table 1 whenever the numbers allow it (and often when they don't), including choosing Multiple Groups rather than Repeated Measure; except for *price*, which is an exception, being frequently used. There is also a universal preference for partition rather than quotition stories; when both are admissible, hardly any quotitions are constructed. (Bell, Fischbein and Greer, 1984.)

Extending the Classification: Change of Size and Mixture Problems

Change of size situations, which include problems about maps and scale drawings, involve the comparison of two related objects, which may be measured in the same or different units. It might be assumed that in these cases the natural multiplier is the scale factor; but our previous work suggests that at least in cases where the two objects are measured in different units, the within-measure relation might be preferred. Our recent experiments show that this is indeed so. When the same units are used, the multiplicand is the measure of the original object, and the multiplier is the scale factor, but with different units, the roles are reversed. (Bell, Greer, Grimison & Mangen, in press.)

Mixture problems constitute another type. Again, the two constituents of each mixture may be measured in the same or different units. We know that when both are in the same units (eg. spoonfuls of lemon and of sugar) there is a strong tendency to work within mixture (Karplus et al., 1983). Our experiment shows that with different units, the within-measure relation is preferred.

Comparison of Estimation and Choice of Operation Forms

In our most recent experiment, half the students were required to respond by choosing the operation, the other half by making an estimate of the answer. Estimating the answer was easier than choosing the operation for division, and for multiplication by a number less than 1. For multiplication by numbers substantially greater than 1, choosing the operation was easier. We hypothesise that the easier estimates arise because they can be made by a semi-qualitative comparison of sizes; the harder ones demand explicit recognition of the operation so that multiplication table knowledge can be drawn upon.

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CHILDREN'S PERCEPTIONS OF MULTIPLICATION ACROSS PICTORIAL MODELS

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Children's perceptions of multiplicative structure in pictorial situations were examined in forty-eight interviews. Children made mathematical and cognitive distinctions among representations. Responses indicated the availability of a variety of representations for multiplication, with a median of 3.5 representations for each picture.

Multiplication and related multiplicative structures become increasingly important to children's mathematical development in the middle grades. The shift from additive structures to multiplicative structures precipitates, as Hiebert and Behr (1988) noted, a change in the nature and representation of number. With the onset of multiplicative concepts, the nature of the unit becomes associated with multiple entities (Steffe, 1988). Understanding of multiplicative structures requires another critical understanding - a change in referents of numbers (Schwartz, 1988).

The fact that these conceptual understandings are considerably important is evident in the substantive research on dimensional analysis (Vergnaud, 1983; Schwartz, 1988), mathematical constructs and numerical values (Bell, Fischbein, & Greer, 1984), and textual analysis (Nesher, 1988).

Within a problem solving context, Nesher substantiated the importance of textual analysis and imagining a situation. In contrast to word problems, where the situation has to be imagined, we were interested in observing what children understand about multiplication from situations which are explicit. We suggest that, for any child, a variety of visual models for multiplication may have been developed in cognitive schemes. When presented with situations and questions requiring application of some multiplicative scheme, the child matches internal and external representations, based on his conceptual requisites. Resulting from this matching, one or more of these internal representations are instantiated.

As an initial investigation of this aspect of multiplicative scheme, we first tried to identify which internal models are available to children and for which of the designated situations they apply any of these models. Specifically, an interview was designed using static pictures (Beattys & Maher, 1989). Consistent with Lesh's description (1988), these pictures were chosen to allow the child to clarify ideas, to elaborate on aspects of the picture that illustrate, in this case, multiplicative relationships. In this context, certain questions were considered:

Given a range of situations visually different in terms of multiplicative representation, do children perceive any of them as related to particular multiplicative models?

In cases where they identify a picture as multiplicative, are they able to relate elements of the picture with appropriate referents of numbers?

This paper will address these questions by describing patterns and variations in children's multiplicative interpretation across pictorial representations.

Design

Forty-eight interviews were conducted with fourth through sixth grade children (9-12 yrs.) from a cross section of an urban population so that children from various socioeconomic and ethnic groups were represented.

The Interview Five models, or representations, for multiplication (area [A], Cartesian product [CP], discrete array [DA], equal grouping [EG], and linear [L]) were pictorially represented using ten pictures. Two other pictures were included as distractors. (See Figure 1.)

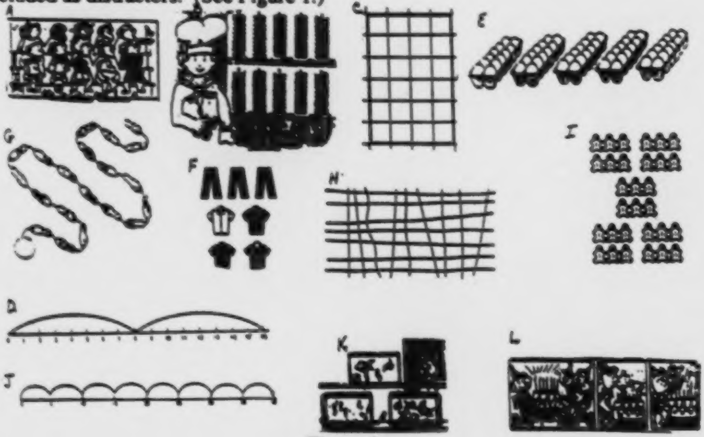

Figure 1

The format of the interview required the child to: pretend to be a teacher who wants to explain multiplication to second graders, select pictures appropriate for teaching multiplication, and then relate these to any symbolic notations they make. For the pictures that were not selected, the child gave a reason for rejecting it.

Based on analysis of responses obtained in a pilot study, a coding sheet was designed in a matrix format so that for each picture, responses were noted in the cell(s) corresponding to the type of multiplicative model the child identified. Descriptors of a range of responses were defined for each model and classification was based on children's responses which were written, spoken, or motioned. Additionally, the authors agreed on those models which were considered appropriate for each picture. Every interview was videotaped and coded data from the matrices were transferred to tables to allow for analyses of general trends as well as individual patterns of response.

Results

Responses indicate the availability of a variety of internal models for multiplication. Table 1 shows the range of children's correct representational responses to pictures by multiplicative model. The appropriate models for each picture were boxed in this table.

Generally speaking, children's match between pictures and appropriate models was a good fit. Overall, the success rates varied between 0 and .93, with a median of 3.5 representations for each appropriate picture.

Of the pictures identified for multiplicative structures, the egg cartons picture was identified with the fewest (2) representations as well as the greatest number of appropriate explanations of multiplication. Three models were suggested for the footprints, gingerbread cookies, and the donuts pictures; there were four models suggested for the remaining seven multiplicative pictures.

The use of equal grouping and repeated addition models extended to nearly all multiplicative situations. As anticipated, equal grouping was the predominant mode of representation in overall analysis. However, children's responses indicated that they made mathematical and cognitive distinctions between repeated addition and equal grouping models. Repeated addition was suggested with only symbolic references and offered as an alternative to other models including equal grouping.

Table 1 Children's Correct Representational Responses by Pictures

Pictures	Grade 4 (20)						Grade 5 (14)						Grade 6 (14)					
	RA	EG	DA	A	L	CP	RA	EG	DA	A	L	CP	RA	EG	DA	A	L	CP
16 # line	.10	.05	0	0	.45	0	0	.07	.07	0	.35	0	.14	.36	0	0	.21	0
5 # line	0	.05	0	0	.50	0	.14	.21	.07	0	.50	0	.07	.21	0	0	.36	0
footprints	.05	.35	0	0	0	0	.14	.43	0	0	.14	0	0	.57	0	0	0	0
gingerbread cookies	.10	.75	0	0	0	0	.07	.57	.07	0	0	0	.21	.86	0	0	0	0
eggs	0	.70	0	0	0	0	.21	.63	0	0	0	0	.21	.93	0	0	0	0
donuts	.20	.65	0	0	0	0	.14	.57	.14	0	0	0	0	.71	0	0	0	0
crowd	0	.40	.35	0	.05	0	.07	.43	.14	0	0	0	.07	.71	.14	0	0	0
parallel lines	.05	.05	.25	.25	0	0	0	.14	.07	.14	0	0	0	.43	.28	.14	0	0
wavy lines	0	0	.20	.05	0	0	0	.21	.07	0	0	0	.07	.07	.21	.07	0	0
candles	0	.15	0	0	0	0	0	.21	0	0	0	0	.07	.21	0	0	0	0
fish	0	0	0	0	0	0	0	.14	0	0	0	0	0	.14	0	0	0	0
clothes	.05	0	0	0	0	.20	0	0	.07	0	0	.29	0	0	0	0	0	.79

With different scales, distinctions were made in many of these children's linear interpretations of number line. For the 16 number line, a reoccurring response was to identify its use with addition.

Pictures included as distractors were often altered to allow for correct application of multiplicative elements. In cases where this adjustment occurred, the modification enabled all but one child to produce an equal grouping model for multiplication.

Overall, the differences among grade levels were minimal. The greatest variability in responses was suggested by fourth graders. Sixth graders demonstrated the most consistent use of one representation, equal grouping. The greatest disparity among grade levels was for the Cartesian products; sixth graders were significantly more successful than children in earlier grades.

In terms of the referents of numbers, children were most successful in identifying ExI situations [See Note 1] with equal grouping representations. Many children seemed to recognize multiplicative situations with the number line and array models, but had trouble connecting written symbols with referents in the picture. A few children successfully identified the two ExE situations, area and Cartesian products.

Table 2 shows the individual performance in terms of the representations applied by five students. In this table the number of correct responses for a given model are identified. Pictures which were identified correctly with more than one model are underlined and included in totals.

Table 2 Summary of Individual Representations for Pictures

Name (Grade)	No. of Reps.	RA	EG	DA	A	L	CP	Pictures (Correct)
Gilbert (4)	5	3	4	1	1	2	-	ABCDE IJ
Jack (5)	4	-	3	7	1	2	-	ABCDEFGHIJ
Ellen (5)	4	5	6	-	-	1	1	AB EFG IJ
Robert (5)	2	-	1	-	-	3	-	DE G J
Verna (6)	3	1	7	-	-	-	1	AB DEFG IJ

Responses of children identified in Table 2 illustrate the variations in responses when individual performances are considered. Childen such as Jack, Robert, and Verna were most successful in recognizing multiplicative structures in situations in which they could apply a particular scheme. The same is true for Ellen, but to a lesser extent. Though it was not suggested in the protocol, Gilbert and Ellen offered multiple approaches [representations] for presenting a few of these pictorial situations. This again is true for Jack and Verna, but in each of these cases the multiple representations always included their dominant representational model. Verna's responses were, in a sense, typical of responses recorded for most other sixth graders - equal grouping was her predominant representational model.

Discussion

If we consider problem solving as pertaining to the interpretation of the problem, the representation of the situation, and the procedures applied to both, then it becomes apparent why it is difficult to isolate the child's representation. Further, children are often unaware of their own representations of the situation. This paper describes results of interviews developed to explicitly study children's application of multiplicative structures to representations of situations.

Children made mathematical and cognitive distinctions among pictorial situations. There is evidence that, for most of these children, a variety of representations exist, and when presented with a situation, they select one representation which they consider appropriate for the situation. Many children's correct representational responses were consistent with the structure of the pictorial situations. Some children repeatedly demonstrated representations that did not correspond to the structure of the pictorial situation, but rather related to an apparent internal representation.

Results indicate great variation in the ways that children interpret pictorial situations. While a sampling of children demonstrated the availability of more than one scheme for multiplication, a more detailed look at individual response patterns might indicate more clearly how internal representations influence a child's ability to attend to a multiplicative situation. Work is in progress to address this issue as well as other related issue.

Notes

Kaput (1986) abbreviates the definition of Schwartz's notation as follows: extensive quantities denote the existence of a particular entity; intensive quantities can be understood as rates. 'ExI' is extensive quantity times intensive quantity; 'ExE' is extensive quantity times extensive quantity.

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PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF PROPORTIONAL REASONING MISSING VALUE PROBLEMS

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This study examined the quality of preservice elementary teachers' understanding of missing value proportional reasoning word problems. This report focuses on teachers' understandings of solution strategies, including their ability to explain these strategies conceptually (why it works) rather than merely procedurally (how it works). Many of the preservice teachers in this study had only a procedural understanding of these problem situations.

Researchers have examined adolescents' achievement and use of solution strategies (for example, Karplus, Pulos, and Stage, 1983a & b; Bezuk, 1986) on proportional reasoning word problems. These investigations have recently begun to be extended to preservice and inservice elementary teachers (Bezuk, 1988). These findings indicate that the type of numeric ratio affects teachers' use of solution strategies in solving proportional reasoning problems, especially their use of the unit rate and factor of change strategies and the cross multiplication algorithm. Also, many elementary school teachers have more than one solution strategy in their repertoire and are able to examine problem characteristics and choose an appropriate strategy based on those findings. Teachers' understanding of the strategies they use is another important component of their ability to effectively teach these strategies. This research examined the quality of teachers' understanding of various methods of solving proportional reasoning word problems and these teachers' ideas on teaching this topic.

Shulman (1987) noted that research-based knowledge of teachers' understanding of content is at the very heart of his definition of needed pedagogical content knowledge. Pedagogical content knowledge is "the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students (Shulman, 1987, p. 15). It also includes an understanding of factors affecting the learning of specific topics and includes "knowledge of the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations--in a word, the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9).

Pedagogical content knowledge of proportional reasoning includes (but is certainly not limited to) the following: (1) awareness of types of problems, such as missing value and comparison problems, and the varying degrees of difficulty of each type, (2) awareness of types of problems settings or contexts, such as buying, speed, mixture, density, and consumption, (3) awareness of and ability to use various solution strategies for solving proportional reasoning word problems, such as those listed in Table 1, (4) awareness of the relationship between the type of numeric ratio (see Table 2) and the ease of use of various strategies (e.g., Bezuk, 1988), (5) methods of including these topics in instruction, and (6) awareness of common student misconceptions (e.g., Hart, 1981). The study reported here provides more information regarding the degree and quality of the pedagogical content knowledge of prospective elementary teachers.

METHOD

The subjects of this study were 20 preservice elementary teachers who were midway in a fifth-year teacher preparation program. These preservice teachers had recently completed all but one hour of a forty-eight-hour-long mathematics methods course taught by this investigator. Proportional reasoning had not been discussed in class; methods for teaching proportional reasoning were covered in the remaining hour of the course which was held following the completion of the interviews described herein. Each subject took a 24-item written test of proportional reasoning ability, each problem containing one of four types of numeric ratios and using a buying context.. The four types of numeric ratios were integral ratios both between and within rate pairs (referred to as "Both Integer"), integral ratios within rate pairs ("Integers Within"), integral ratios between rate pairs ("Integers Between"), and no integral ratios ("Non-integer"). These categories are those used by Karplus et al. (e.g., 1983b) throughout their work. Table 1 illustrates the problem setting and types of numeric ratios used in this study.

Table 1 Types of Numeric Ratios

Sample problem: Ann and Kathy each bought the same kind of bubble gum at the same store. Ann bought 2 pieces of gum for 6 cents. If Kathy bought 8 pieces of gum, how much did she pay?

1. Both Integer:
   2 pieces for 6 cents 8 pieces for how much? (24 cents)

2. Integers Within Rate Pairs:
   2 pieces for 6 cents. 5 pieces for how much? (15 cents)

3. Integers Between Rate Pairs:
   2 pieces for 3 cents. 8 pieces for how much? (12 cents)

4. Non-Integer:
   6 pieces for 10 cents. 9 pieces for how much? (15 cents)

Scoring of the tests included the classification of the strategy used to solve the problem. These data are reported elsewhere (Bezuk, 1988).

These teachers were then individually interviewed in order to gain greater insight into the solution strategies used and their understanding of proportional reasoning situations. The interviews consisted of several parts: the subjects explained how they solved several different problems from the written test, were shown at least three other methods for solving the same problems and were asked to evaluate these methods, and were asked to solve anther problem in as many ways as they could. These student teachers were also asked several questions about teaching proportional reasoning to children: what parts might be harder or easier for student to understand, what parts might be harder or easier for them to teach, what manipulatives, problem settings, and/or activities they might use to teach this topic, and what solution strategies (and in what order) they would present when teaching this topic. The solution strategies shown are an expansion of the strategy categories used by several other researchers (Heller et al., 1985, Karplus et al., 1983b, and Noelting, 1980). Table 2 presents examples of each of these seven strategy types.

Table 2 Examples of Each Strategy Type

Example problem: Ann and Kathy each bought the same kind of gum at the same store. Ann bought 2 pieces of gum for 6 cents. If Kathy bought 8 pieces, how much did she pay?

1. Unit rate
   Each piece costs 3 cents, so 8 pieces will cost 24 cents.
   8 pieces * 3 cents/piece = **24 cents**
2. Factor of change
   Kathy bought 4 times as much gum as Ann, so Kathy should pay 4 times as much.

   4 * 6 cents = **24 cents**
3. Cross multiplication algorithm

$$\frac{2\text{ pieces}}{6\text{ cents}} = \frac{8\text{ pieces}}{x\text{ cents}}$$

$$\begin{aligned} 2\text{ pieces} * x\text{ cents} &= 6\text{ pieces} * 8\text{ cents} \\ 2x &= 48 \\ x &= \mathbf{24\ cents} \end{aligned}$$

4. Generate pairs
   2 pieces for 6 cents
   4 pieces for 12 cents
   6 pieces for 18 cents
   8 pieces for **24** cents
5. Equivalent fractions

$$\frac{2}{6} * \frac{4}{4} = \frac{8}{24}$$

6. Equivalence class

$$\frac{2}{6} = \frac{4}{12} = \frac{6}{18} = \frac{8}{24}$$

7. Additive
   Ann paid 4 cents more than the number of pieces she bought, so Kathy must have paid 4 cents more than the number of pieces she bought.
   2 + 4 = 6 cents, so
   8 + 4 = **12 cents**. (incorrect strategy)

RESULTS

Evaluation of Strategies

Subjects were shown examples of problems solved by seventh grade students using various solution strategies. These problems were the same problems that the student teachers solved on the written test. The following section describes the student teachers' analyses and opinions of these strategies.

Half of the twenty subjects commented on the **unit rate** method. Their comments were very positive: "It's easiest to understand"; "Unit rate is easier. Good way to do it also."; and "That's fine. It's easy to do the mental math required to do it. It's real obvious.".

Fourteen subjects commented on the **factor of change** method. This strategy was not as well-received as the unit rate method. The following are some of the comments: "Wow- very interesting. It works. When numbers get messier, it would be a lot more confusing."; "For heaven's sake. My goodness. How old was this child? Did they get special training?"; "A little confusing; a longer process--though only 2 steps, it seems harder to explain."; and "I don't even know if I understand what she did. It doesn't seem to me as clear as unit rate."

Twelve subjects commented on the **cross multiplication algorithm**. This strategy was not as well-received as the unit rate method. The following are some of the comments: "It's easiest to comprehend."; "It's more efficient; not as many things to think about."; "This will eventually help them in algebra. I would use this. This is foolproof."; "I would have never thought of this one. I don't think kids could understand this."; and "I see them getting confused when you flip the reciprocal. I like equivalent fractions better than cross multiplication algorithm."

Four student teachers thought the cross multiplication algorithm was an incorrect method. One student teacher commented that it "works for some numbers but not for all." She felt that there was "no need to introduce this method."

Eighty percent of the subjects commented on the **generate pairs** strategy. Comments were bimodal. Several subjects (9 out of 16) commented on the limitations of this method: "Very, very confusing--too many steps." and "Generate pairs is time-consuming. Throw this out.". However, several other subjects (7 out of 16) made very positive comments: "Makes sense. I can see how kids make lists."; "This is excellent for teaching. When children see a pattern, the concept is easier to use." ; and "It's a lot like making equivalent fractions.", recognizing that fractions formed from the pairs generated are equivalent.

Sixty percent of the subjects commented on the **equivalent fractions** strategy. Opinions of his strategy varied more than any other strategy. Some student teachers thought it was wonderful (e.g., "quickest, fastest , easiest, and fewest calculations." Some thought it was adequate but cumbersome (e.g., "makes sense but I think it's more work setting it up this way than the other way. (unit rate)." Other students were concerned about the general difficulty of fraction concepts (e.g., "this would be more difficult; fractions are harder because it deals with equivalence.").

Most subjects (18 out of 20) commented on the **additive** strategy, which is an incorrect strategy. All but two of the subjects stated that this method was wrong. The remaining subjects either weren't certain that the method was incorrect or did not express themselves well (e.g., "kind of like what I do but I multiply.: and "I can't figure it out. I can't explain it.").

Understanding of the cross multiplication algorithm

Student teachers were asked how they would explain to students **why** the cross multiplication algorithm works. Table 3 shows categories of their responses.

Table 3: Explanations of why the cross multiplication algorithm works

Fractions	7
Algebraic	3
Rote, procedural manner	4
Don't know; no idea	6

Several student teachers saw a connection with fractions, especially to equivalent fractions: "We're setting up equivalent fractions. What I do to the top I have to do to the bottom."; "They're equivalent fractions, but I don't know why."; and "When numbers are multiplied cross ways, they equal each other. This would be a good check for equivalent fractions." Subjects who used an algebraic-type explanation referred on operating on both sides of an equation: "This works because it has to do with one. You have to do the same thing to both sides of the equation."; and "What you do to one side, you have to do to the other. When you do that, it comes out." Other subjects only uses a rote explanation: "I just remember A is to B as C is to D. A x D = B x C. I don't remember the rationale. I was re-taught this over and over.". And other subjects did not know why the algorithm works: "I don't know why, but it was taught that way."; "The teachers' manual doesn't say why."; "I don't know if I know why this works. Kids want to know why and I couldn't explain."; and "The cross multiplication algorithm doesn't teach for meaning. It means something to me but I can't tell you how to do it. Even though both the equivalent fractions explanation and the algebraic explanations could be correct, none of the subjects was able to explain in detail either of these reasons correctly.

DISCUSSION

This investigation revealed that many preservice elementary teachers have only a procedural understanding of proportional reasoning missing value situations and methods of solving such problems. Most of them are unable to explain why the standard method, the cross multiplication algorithm, worked. Many of the student teachers had a vague idea that the reason was related to either equivalent fraction or algebraic notions, but none could piece it together. One wonders how these teachers will be able to teach effectively when they are still struggling with their own understandings.

It is the opinion of this researcher that teacher education programs need to focus on developing understanding of concepts in prospective teachers. Merely being able to produce a correct answer is not adequate, if these teachers are then expected to go into the classroom and assist students in learning concepts as well as procedures. It seems clear that the development of an understanding of mathematics concepts must be a priority of faculty involved in teacher preparation.

The influence of the factors discussed herein on teachers' classroom performance in teaching proportional reasoning and ultimately on their students' performance is a question of interest which warrants further study.

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USING RESEARCH IN RATIONAL NUMBER LEARNING TO STUDY INTERMEDIATE TEACHERS' PEDAGOGICAL KNOWLEDGE

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Intermediate teachers' understandings of rational number concepts are examined using the "practical argument technique". This involves reconstructing teachers' reasoning as a series of premises whose conclusions is an action or intention to act. This technique is illustrated by studying a teacher's response to a question about how a student would compare 3/4 and 3/5. The practical argument analysis shows in what way the teacher's reasoning is or is not in harmony with what is known about the learning of rational numbers.

The goal of this study is to use results from cognitive studies of rational number learning as a basis for examining intermediate teachers' pedagogical knowledge. More generally, the goal is to "reap some fruits" from studies of the learning of mathematics to help better understand the teaching of mathematics (Romberg & Carpenter, 1986). The research base for the learning of mathematics will be results from the Rational Number Project, abbreviated RNP (Behr, Lesh, & Post, 1979-). Teachers' understanding of fundamental rational number concepts such as order and equivalence, their notions of the concept of unit, their facility in using representations in discussing rational number ideas, and their ability to form a coherent, rational explanation all fall within the scope of the inquiry. This paper will describe the framework for relating the research base of the RNP to teacher's knowledge and will illustrate this framework using one example.

THEORETICAL FRAMEWORK

A basic assumption underlying this inquiry is that a knowledgeable teacher knows how her subject matter is learned (Dewey, 1902, 1916; Shulman, 1987). Dewey distinguishes the logical ordering of a subject matter from the psychological nature of its genesis and argues that teachers must be intimately acquainted with the latter. For the teacher, the subject matter "needs to be psychologized; turned over, translated into the immediate and individual experiencing within which it has its origin and significance" (Dewey, 1902, p. 22, Dewey's italics). In more modern language, Shulman speaks of the teacher as knowing how to interpret, express, or represent the subject matter in a way that students can understand it (Shulman, 1987). This notion of a "psychologized" subject matter will be used as an ideal for conceptualizing teachers' knowledge of a content area.

Cognitive research in subject areas such as science and mathematics has lead to several results as to how these subject matters are learned (Resnick, 1983). These results might be used to "psychologize" teachers' knowledge of these subjects and improve the teaching of science and mathematics. At the National Center for Research in Mathematical Sciences Education, work is underway to use knowledge as to how children learn addition and subtraction concepts to build up elementary teachers' pedagogical content knowledge (Carpenter et. al., 1988; Peterson et. al., in press). A knowledgeable teacher, in this approach, would use what is known about the learning of addition and subtraction concepts in the instruction of her children. Initial results have indicated that student achievement is greater for those teachers whose practices and beliefs are in greater harmony with the research position (Peterson et. al., in press).

Though much is known about the learning of school subjects, by and large this research knowledge is in the hands of specialists such as psychologists and mathematics educators rather than classroom teachers. A model for relating research knowledge and the knowledge used by classroom teachers has been explained by Green (1976) and Fenstermacher (1986, 1987). The underlying assumption of this model is that teachers are rational agents, and that therefore their behavior needs to be understood in a way that would make sense to both the teachers and researchers who study teaching. To "make sense" of teachers' actions, Green and Fenstermacher use a device which was first used by Aristotle in his analysis of human behavior. This device is the "practical argument", a syllogism whose conclusion is an action or an intention to act. Fenstermacher gives the following example of a practical argument for a teacher making the decision to organize her classroom according to the principles of direct instruction:

> As a teacher, I want to teach in ways that yield as much student learning as possible.
>
> Well-managed classrooms yield gains in learning.
>
> Direct instruction is a proven way to manage classrooms.
>
> My students and I are together in this classroom.
>
> ACTION: I am organizing my class according to the principles of direct instruction (Fenstermacher, 1986, p. 43).

An example of a practical argument, taken not from research in direct instruction but from research in the cognitive learning of addition and subtraction concepts, might be reconstructed as follows:

> "[C]hildren invent a great deal of their own mathematics and... they come to school with well developed informal systems of mathematics" (Romberg & Carpenter, 1986, p. 853).
>
> "Children's invented strategies for solving addition and subtraction problems are frequently more efficient and more conceptually based than the mechanical procedures included in many mathematics programs" (Romberg & Carpenter, 1986, p. 855).
>
> I want my students to develop structures for solving addition and subtraction problems which are efficient and conceptually based.
>
> ACTION: I attempt to build upon students informal structures when teaching addition and subtraction concepts.

A practical argument is essentially an ideal or limiting case of the type of reasoning that one would expect or hope from any person whose actions are guided by reasons. If a teacher's knowledge can be reconstructed as a practical argument whose premises are based on results from cognitive theory, then one might say that the teacher has psychologized her understanding of the subject matter (cf., Morine-Dershimer, 1987).

METHODOLOGY AND DATA SOURCE

The Rational Number Project (RNP) has provided a map of the psychological terrain of the learning of rational numbers (cf., Post et. al., 1985;

1988). This research shows that the learning of rational number concepts hinges on central ideas, such the concepts of order, equivalence, and unit, as well as the notion that translation among different representations or embodiments of these concepts facilitates understanding (Post et. al., 1985; Lesh et. al., 1987). These results from the RNP can be used as a research base for investigating the degree to which teachers' knowledge is "psychologized".

Two hundred thirty eight intermediate teachers from two midwestern sites took a battery of tests in connection with a NSF project to generate profiles of mathematical understanding for teachers (Post et. al., 1988). The questions that were asked follow from previous work conducted by the RNP in connection with students. However, the questions were asked from the point of view of the teacher responding like a hypothetical student or explaining a problem to a hypothetical student. In one task, for example, teachers were asked how students would solve a series of ordering problems that involved rational numbers. For example, teachers were asked how students would answer the questions: "What is larger, 3/4 or 3/5". Then, in later questions, teachers were confronted with erroneous responses that students might make to the comparison problem and then asked how they would respond. In the preceding example, teachers were asked to respond to a student who argued: "3/4 is less than 3/5 because there are less pieces".

A PRACTICAL ARGUMENT ANALYSIS OF ONE RESPONSE

When asked how a child who understood fractions would compare 3/4 and 3/5, one teacher responded as follows:

> They'd work it out in equivalent fractions. They would multiply. The numerators are the same and they know if you multiply 3 by a higher number [5, it] would be larger than 3 times 4.

One of the main assumptions that the teacher appears to be making is that children who understand rational number concepts would compare 3/4 and 3/5 by finding a common denominator. What is of interest are the implications of this assumption for the teaching of mathematics. How might this assumption figure into a practical argument that would guide the teacher's actions?

It seems plausible that this teacher's actions could be guided by the following argument:

1. I want my students to "understand" how to compare fractions.
2. Students who "understand" how to compare fractions use the common denominator procedure.

Action: I teach my students the common denominator procedure for comparing fractions.

This is not to say that the teacher would actually engage in this linear reasoning in the classroom. The point is that this practical argument might be used by the teacher to "make sense" of a teaching decision, if she were asked to do so. Premise 2 would be the cornerstone for a teaching decision.

Premise 2 can be examined more closely for a possible fit or misfit with the research base generated by the RNP. It is possible to imagine a hypothetical situation where this premise would be very reasonable. For example, if human beings thought within a computer language such as BASIC, then it might make sense, pedagogically, to equate "understanding how to compare fractions" with "using the common denominator procedure". A BASIC program that compared fractions using the common denominator procedure would be simple, relatively fast, and efficient. A teacher might be well advised, in this hypothetical case, to "program" the common denominator procedure in her students.

However, a BASIC common denominator program would do a poor job of simulating the way many children compare fractions. Children observed in the RNP, for example, would often solve this or similar comparison problems by noting that, for a given unit, 4ths are larger than 5ths. Therefore 3/4 is larger than 3/5 (cf. Behr & Post, 1987). Premise 2 indicates a reliance on a formal procedure (finding common denominators) when a less formal observation (cutting a unit into four pieces provides larger pieces than cutting the same unit into five pieces) would suffice. It could be argued that premise 2 is false, if the notion of "understanding how to compare fractions" is the same as that held by many mathematics educators.

The conclusion of the practical argument is also problematic. Research conducted by the RNP has found that children often do not have a workable concept of rational number size. Though the common denominator procedure will always yield two fractions that can be easily compared, this algorithm does not necessarily promote an understanding of the size of the rational numbers. Performing the procedure becomes a substitute for thinking about (or estimating) the relative sizes of the rational numbers. Put another way, "an interest in the formal apprehension of symbols and their memorized reproduction" becoming "a substitute for the original and vital interest in reality" (Dewey, 1902, p. 28). Overreliance on these formal procedures instills in the child a mindless, factory-assembly mentality, takes away from her the spirit of a democratic education, and promotes in her the idea that mathematics is something mysterious whose power resides in authority rather than in her inventive powers (a paraphrase of Dewey, 1902; 1916, Ch. XIV).

CONCLUSION

The research base generated by the RNP can be used to examine and conceptualize the pedagogical content knowledge of intermediate teachers. Reconstructing teachers' responses to certain tasks as practical arguments can show to what extent these teachers have a "psychologized" understanding of their content area. It is hoped that this line of inquiry might lead to a better understanding of ways to relate the learning of mathematics with its teaching.

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THE HISTORICAL DEVELOPMENT OF LOGARITHMS AND IMPLICATIONS FOR THE STUDY OF MULTIPLICATIVE STRUCTURES

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This investigation into the historical traditions that led up to the invention of logarithms has offered new evidence of conceptions of multiplicative structures based on proportional change which are independent of properties of addition. This work should enrich and challenge current conceptual work on the development of concepts of multiplication in elementary and middle school grades. A longer version of this paper is available from the authors.

INTRODUCTION

Research in the area of multiplicative structures has typically modelled multiplication either as a form of repeated addition (e.g. Fischbein, 1983) or of simple proportion (e.g. Vergnaud, 1988). In both cases the relationship between variables is linear, and thus additive, i.e. additive changes in one variable correspond to additive changes in the other. Thus, we would claim that both models are built on an additive model of multiplication. Confrey (1988, 1989) has conjectured that in many situations modelled as repeated multiplication, e.g. exponentially or logarithmically, multiplication is a primitive operation that does not neet to be built from additive models. These multiplicative structures, which she has called splitting structures, can be established independently of counting structures and allow the density of the rational numbers to be constructed beginning with the unit. This conjecture has motivated our interest in the history of geometric series and the development of logarithms and it is within this framework that this historical study contributes to our understanding of multiplicative structures.

THE ROLE OF REPRESENTATIONS

An important aspect of this investigation involves the changing nature of mathematical expressions which has occurred during the last four hundred years. Much of modern mathematics is dominated by concepts of function where one variable is typically expressed as a mathematical "function" of one or more others. For example, in a situation where population size, P, is modelled as an exponential function of time, t, the standard functional representation would be $P=P_0a^t$. This relationship might, however, also be represented in terms of how P changes for fixed time periods: $P_n=aP_{n-1}$, emphasizing the constant multiplicative rate of

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change; or it may be represented in ways that emphasize the isomorphic relationship between population and time, i.e. additive changes in time correspond to multiplicative changes in population: $[(t_a - t_b)=(t_q - t_r)] <=> [(P_a/P_b) = (P_q/P_r)]$. Although the development of the function concept is often seen merely as a natural progression allowing computational and theoretical advances in mathematics, it also can encourage a particular way of looking at and solving mathematical problems.

TRADITIONS LEADING TO NAPIER

From the time of Aristotle through the Middle Ages, mathematical theories had been built on proportional relationships between various quantities, rather than functional equality. The use of proportional relationships between structures places the emphasis on a comparison of how change occurs within structures, with little concern for correlating values between structures. Thus the equation $A = kr^2$ is only useful if it is known that $k=\pi$, for that is the value that allows one to predict area from a given radius, or vice-versa. In the proportional relationship, $A_2:A_1 :: r_2^2:r_1^2$, there is no correlation between values, but only between how changes in area correspond to changes in the (square on the) radius. Likewise, $P=P_0a^t$ is useful only if one has a value for 'a' and a value for P_0. The proportional (or isomorphic) form, $[(t_n - t_m)=(t_q - t_r)] <=> [(P_n/P_m) = (P_q/P_r)]$, is a statement that additive changes in time correspond to multiplicative changes in population, and is independent of any particular relationship between a value of t and a value of P.[1]

Geometric and Arithmetic Series

Since the time of Archimedes, it had been recognized that when arithmetic and geometric series were juxtaposed, addition in the arithmetic series corresponded to multiplication in the geometric series. Stifel, in the sixteenth century, was apparently the first to extend these series to include fractional values in the geometric series with corresponding negative values in the arithmetic series (Smith, 1915, p.86). Thus one of Stifels tables was:

-3	-2	-1	0	1	2	3	4	5	6
1/8	1/4	1/2	1	2	4	8	16	32	64

---

[1] Thus transformations on a structure that do not change the relationship between elements in the structure will have no effect on the proportional representation.

Although mathematician's working in this tradition did everything but invent log tables (Smith, 1915), they were hindered in this latter task by two factors: first, by concentrating on geometric series, they always defined a system in terms of discrete powers of a specified ratio, and thus, second, had no way of visualizing the process by which the entries in the geometric series would be space at equal arithmetic intervals, an essential component in creating a true log table. Thus, in this tradition, Burgi, a contemporary of Napier, published extensive tables of powers of 1.0001 (Whiteside, 1969). Since his intent, that of making calculation easier, was the same as Napier's, he is often considered a co-inventer of logarithms, but his apparent lack of the concept of continuity in the geometric world, thus preventing equal spacing of entries in his geometric table, was conceptually far different from that of Napier.

The World of Ratio: A Multiplicative Continuum

In the thirteenth and fourteenth centuries, two mathematician's, Thomas of Bradwardine (Crosby, 1961) and Nicole Oresme (Grant, 1966), working on Aristotelian problems relating force to velocity, posited a world in which ratios were the primary entities, and multiplication the primary operation. Oresme defined this world as an exact replica of the world of numbers as primary entities with addition as the primary operation. Whereas today, we would talk about the isomorphism between these two worlds, Oresme considers their properties almost inherent. Thus in considering ratios, the only possible operation is multiplication. Thus "twice" the ratio 3/4 is equivalent to applying the ratio 3/4 twice, or twice 3/4 is $(3/4)*(3/4) = 9/16$. Likewise, 3/4 is "half" of 9/16. Likewise to have a ratio operate on another ratio (a ratio of ratios) is defined exponentially. As a contrast, if the ratio 3/4 operates on the number, 16, one divides 16 into 4 arithmetic means (each equal to 4), then adds three of them together to get 12, or 3/4 of 16 is $(3/4)*16$, whereas if the ratio 3/4 operates on the ratio, 16/1, one first divides 16/1 into 4 geometric means (or takes the fourth root), 2/1, then multiplies it together 3 times. Thus 3/4 of the ratio 16/1 is 8/1, or 3/4 of 16/1 is $(16/1)^{3/4}$. Oresme defines part, parts, commensurable, and incommensurable for the multiplicative world of ratio exactly parallel to Euclid's definitions of these terms for number in Book VII and magnitude in Book V of the Elements. In the end, he has constructed a world of ratio independent of counting structures, in which ratio forms a continuum, and, in which, a full set of operations for ratios have been defined, parallel to those for numbers.

JOHN NAPIER AND THE INVENTION OF LOGARITHMS

Although we do not know how familiar Napier was with the work of Oresme or Stifel, in many ways his work seems like a marriage between Oresme's world of continuous ratios and the developing work on correspondence between arithmetic and geometric series. To some, these worlds undoubtedly seemed incommensurable. How could a correspondence, a discrete notion, be combined with continuous quantity? Napier's great invention, which allowed this union, was the creation of a geometric model in which two lines would be constructed in such a way that additive change in one line would corresponds with proportional change in the second. He specifies that two points will start moving at the same time and with the same velocity. One point , the arithmetic point, will continue to move at that velocity continuously. The second point, the geometric point, starts on the circumference of a circle[1] with radius 10^7, and moves towards the center such that the distance it covers in equal time periods will always be proportional to its distance from the center.

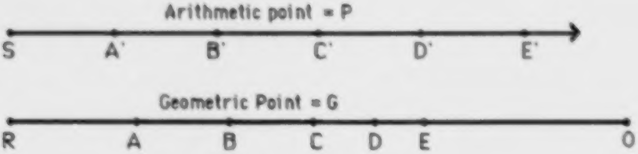

Figure 1

The top line represents the motion of the arithmetic point which has started at S and moves from S to A', from A' to B', from B' to C', etc. in equal increments of time. The bottom line represents the motion of the geometric point which has started at R and moves towards the center of the circle, 0. This point moves from R to A, from A to B, from B to C, etc in the same equal increments of time as the arithmetic point. Thus the length of R0 = 10^7 and the ratio of movement to distance from 0 (in a fixed time) is constant, thus RA:R0 = AB:A0 = BC:B0 = CD:C0 = ...

Napier defines logs as follows: The log of the distance, G, of the geometric point from the center after any amount of time is equal to the distance, P, the arithmetic point has travelled in the same amount of time. Thus, In Figure 1, log(0R) = 0; log(0A) = SA'; log (0B) = SB', etc. Thus unlike modern log tables, G decreases as P increases. The great insight of Napier was that he need not specify the ratio of change of position of the geometric point, nor specify the units of time. In any equal time periods, the distance moved will be proportional to the distance from the center. In fact, since no discrete rate of change is specified, it is not possible, in Napier's world, to juxtapose an arithmetic and geometric series in the same manner as Stifel and Burgi (see Figure 2). It also makes no difference what the initial velocities of the two points are, as long as they are the same.[1] In effect, Napier has created a continuous geometric world and juxtaposed it with a continuous arithmetic world.

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[1] Napier framed his problem within a world of sines of right triangles inscribed in a circle of radius 10^7. The reasons for this are unimportant to our present discussion

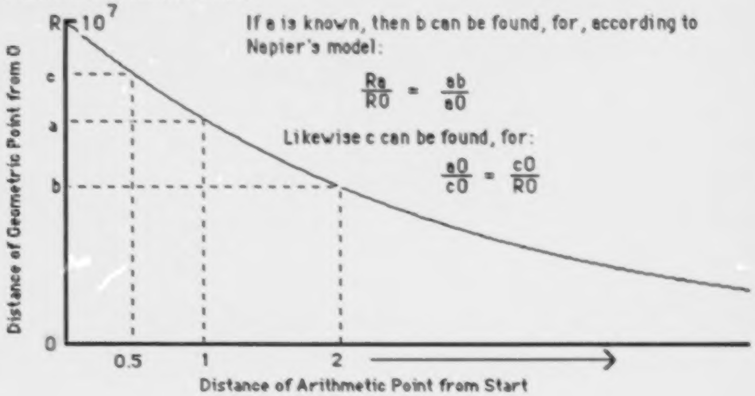
$R=10^7$
If a is known, then b can be found, for, according to Napier's model:
$\frac{Ra}{R0} = \frac{ab}{a0}$
Likewise c can be found, for:
$\frac{a0}{c0} = \frac{c0}{R0}$
c
a
b
0
0.5
1
2
Distance of Geometric Point from 0
Distance of Arithmetic Point from Start

Figure 2

In Figure 2, if the ratio of change of the geometric point per unit of travel by the arithmetic point (or per unit of time) were given, then the point (1,a) could be found, for "a" would be calculated by applying the specified ratio to 10^7. Once "a" is known, as many additional equally spaced points on the arithmetic scale as desired can be added by taking an appropriate number of geometric means on the geometric scale. This was the model of Stifel and Burgi, but does not create a true log table, since the points on the geometric scale are not equally spaced.

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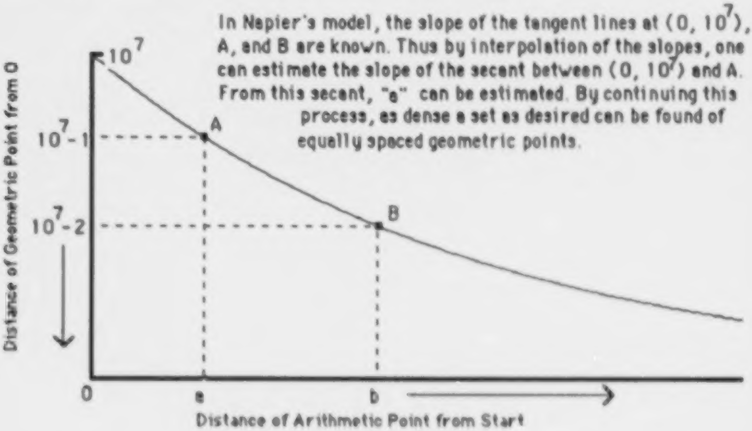
[1] Napier did not think in terms of a "base" for his table. However, for any initial velocity for the two points, whenever the arithmetic point moves one unit of distance,the distance of the geometric point from the center will decrease by a ratio equal to the 10^7 root of 1/e, a number only slightly less than unity. Thus his "base" can be thought of as the 10^7 root of 1/e.

From these initial conditions, Napier made an important observation: "A geometrically moving point approaching a fixed one has its velocities proportionate to its distance from the fixed one." (p.18), i.e. the velocity of the geometric point in the model above will always be proportional to its distance from 0, which, in modern terms, is notationally equivalent to the statement: $dy/dx = ky$! This statement is made more from insight than from what we would call proof, for Napier simply makes the argument that since, by hypothesis, in any finite time period, the point moves a distance proportional to its distance from 0, that the velocity in any finite time period must then be proportional to the distance from 0. This insight gives Napier the ability to specify equally spaced points on the geometric scale, and calculate, as accurately as desired, the matching points on the arithmetic scale, for he now knows the velocity of the geometric point (relative to its initial velocity) at any specified position of G. In a modern interpretation, this means that the tangent to the curve at any point is known, and thus the secant between any two separate points can be accurately estimated.



In Napier's model, the slope of the tangent lines at $(0, 10^7)$, A, and B are known. Thus by interpolation of the slopes, one can estimate the slope of the secant between $(0, 10^7)$ and A. From this secant, "a" can be estimated. By continuing this process, as dense a set as desired can be found of equally spaced geometric points.

10^7
10^7-1
10^7-2
A
B
0
a
b
Distance of Geometric Point from 0
Distance of Arithmetic Point from Start

Figure 3

However, this is still not a log function in the modern sense, i.e. that it obeys the "laws of logarithms". For example, we assume that log (A/B) = log (A) - log (B). In particular, if B=1, then log(A)=log(A/1)= log (A) - log (1), which can only be true if log(1)=0. In Napier's system, log(1) is well over 10^7. Napier's logs force us to think of multiplication as a 'ratio action' rather than a binary operation. Thus in Napier's world the proportional relationship holds: A:B::C:D <=> (log(A)-log(B))=(log(C)-log(D)). Thus to multiply 256 times 3978 is to solve the ratio problem: 256:1::A:3978, and Napier has told us that the difference between log(256) and log(1) is the same as the difference between log(A) and log(3978). Thus log(A)=log(256)-log(1)+log(3978). Likewise to divide 4288 by 333 is to solve the ratio problem: 4288:333::A:1, from which we find that: log (A) = log(4288)-log(333)+log(1).

CONCLUSIONS

In looking at the work of Napier and his predecessors, we see a tradition, going back at least to the Greeks, which saw multiplication as the primary action in certain situations, independent of concepts of addition. Modern concepts of number line and function, and metaphysical concepts of cause and effect have both contributed to a reduction of interest in this area. Thus most modern research on multiplicative structures has overlooked this tradition of multiplication as an independent structure in favor of models based on some form of repeated addition.

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REPRESENTATIONS, METACOGNITION, AND PROBLEM SOLVING

CHILDREN'S REPRESENTATIONS OF ARITHMETIC PROPERTIES IN SMALL GROUP PROBLEM-SOLVING ACTIVITIES

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The problem-solving behavior of eleven seventh-grade children, working together in small groups in a five session teaching experiment within a classroom setting is described. The children were asked to construct solutions to three problem tasks which contained common structural elements but in each case used a different set of concrete nonnumerical elements. Analysis of videotapes of the sessions revealed that a variety of strategies were used to construct solutions, that there was recognition of the relatedness among problem representations, and that students were able to generalize to representations of similar structures using numbers. Analysis of interactions among members of each group indicated that working together facilitated the problem-solving process for individual children.

Theoretical Framework

The importance of structure, both of the concepts being understood and the construction within the experience of the learner of a working model of those concepts, is basic to any theory of learning (Dienes and Jeeves, 1965). One implication of this understanding of mathematics and learning, according to Jeeves and Greer (1983), is the importance of developing an awareness of the structural relationships in the mathematics that is being used and an ability to recognize structural similaritites in situations that appear on the surface to be different. Bruner (1960) advocates learning the fundamental structure of a subject as a means to knowledge that can be retained longer in memory, and transferred more effectively to new learning.

A weakness in much research into the transfer of knowledge, according to Lave (1988), is that it has occurred within a context devoid of social interaction or consideration of factors that motivate problem solving, presupposing that **abstracted knowledge** is the context of problem solving. An alternative approach advocates that consideration be given to learning that arises out of shared activity in which representations of ideas are constructed and discussed (Brown, et al, 1989). This activity is facilitated by children's working together in small groups on mathematical tasks which provide opportunities for communicating mathematically, sharing ideas, developing and revising hypotheses, and defending solutions (Noddings, 1985).

Previous research analyzing small group problem-solving activities provided examples of meaningful learning of particular mathematical structures and rich data for consideration of the process of children's mathematical thinking (Alston and Maher, 1988; Maher, Alston, and O'Brien, 1986).

Objectives

The objective of this study was to describe and compare the problem-solving behaviors of children working in small groups to solve three problem tasks. Each of the tasks offered a different concrete nonnumerical embodiment to be used to construct models of the structure of certain properties of a binary operation on a set of elements: namely, closure, commutativity, identity, and inverse elements. Particular behaviors that were studied were: (1) **construction, monitoring, and revision of solutions** on the basis of the concrete model, imagistic representation in the form of charts or drawings, and/or conceptual understanding; (2) **connections among representations** within and among tasks indicating recognition of structural similarities and differences; (3) **generalization** to mathematical ideas involving numbers; (4) **interactions** among group members and **individual roles** within the group contributing to the development of solutions.

Methods and Procedures

The study was conducted in 7th grade classes in two schools, one independent K-12 and the other a public K-8 elementary school. Five consecutive 45 minute class sessions in each of the schools provided 12 and 13 year old children the opportunity to construct solutions to three concrete nonnumerical problem tasks dealing with the structure of the properties of closure, commutativity, identity and inverse.

The teachers partitioned each class into groups of two or three children for the five sessions on the basis of similarity of ability and potential social compatibility. Within each class two groups were randomly chosen to be videotaped during all sessions. A study of the problem-solving behavior of eleven children, seven boys and four girls, in four groups, two from each school, was the basis for this paper. The two public school groups included one group of two boys (G1) and a second of two boys and one girl (G2). The independent school groups that were videotaped included one group of two boys and one girl (G3) and a second group of two girls and one boy (G4). The members of each group had been a part of the same mathematics class throughout the year and had been accustomed as a part of regular instruction to working in small groups to solve problems.

After the children were seated in their group, a script for each of the problems in turn was given to each child along with sets of the objects appropriate to the task. The children were instructed by the teacher to choose one person to act as official recorder and to have some agreement on the responses recorded. Each child, however, was asked to complete a problem script with his or her own ideas about the solution which might be different. A final section on each task asked the children to reflect on the problem solving and to note (1) what they liked and disliked about it, and (2) what other problems or ideas, if any, were called to mind.

The directions concerning each operation were written as a part of the script and the children were asked to demonstrate understanding of the operation before beginning to construct solutions. The teacher was instructed to respond to questions that sought to clarify understanding of the meaning of each operation but not to intervene as the children constructed solutions. The children were permitted as much time as required to complete each problem task, then returned their problem sheets and materials to the teacher and received the set for the next task.

Transcripts of the videotapes of the five sessions for each of the four groups of children along with observers' notes and the children's work sheets provided data for the analysis.

The Problem Tasks

The Dolls Task: A pair of small figures, a boy and a girl, were used to enact the elements of the set; four 180 degree rotations of these two figures taken together from a facing front position: **Both Turn, Only Boy Turn, Only Girl Turn,** and **Nobody Turn.** The operation of the set was defined as one rotation followed by a second and the result is the single rotation from a facing front position that would leave the figures in the same final position.

The Problem with Cards: A set of five cards, each with a different polygonal shape cut out, constituted the elements of the set with the operation defined as putting one card on top of another and the result being the shape of the hole formed by the two cards together.

The Roads Task: The members of the set, introduced as Road Cards, were index cards each having lines from four equally spaced beginning points on the left side to corresponding end points on the right. The operation was introduced as one Road Card followed by a second and the result was the single card that had the beginning points of the first card and the end points of the second.[1]

Results

Because of limitations in space, the children's mathematical behaviors are illustrated with examples of **problem-solving approaches used to construct solutions to various parts of the problems** and then summarized in Table 1. An example of group interaction as a part of constructing a solution is then described and followed by a summary of **individual children's roles within the groups** in Table 2.

All four groups were successful in **completing the charts** for each of the **three operations** with each group relying extensively on the concrete objects to figure out these results. Members of each of the groups, after confirming that they were correctly performing the operation in each task, also offered results for particular pairs of elements without first using the objects. When this occurred, the result was justified to the rest of the group in one of several ways including: (1) symmetry or other patterns noted in the chart (all groups), (2) function of identity and/or inverse elements (G1, G2, G4), (3) description of geometric shape or transformations (G3), and (4) mental predictions which were confirmed with objects when questioned (G3). All four groups used the **Road Cards** to determine the result of each possible combination for that operation.

Strategies for solving the problems about **closure** varied. All four of the groups referred to the actual objects. G1 and G4 based their solutions about closure on charts. G1 constructed new charts for the **Dolls Task** to determine whether subsets of the set of four elements were closed whereas both groups continually referred to the appropriate section of the charts in each problem. G2 and G3 both based their arguments about closure on the importance of identity and inverse elements with G2 referring to numbers in order to determine the function of each element in the subgroup of two:

C1: **Nobody Turns sort of acts like zero in a problem. It's always the original number with zero because Nobody Turns just leaves it the way it is.**
C2: **It's like only having one command, right?**
C3: **It cancels --- It takes you where you started.**
C2: **No! It stays the same. It doesn't take you where you started.**

Based on this discussion, G2 was successful in determining that Nobody Turns and Only Boy Turns form a closed set, however they failed to realize that the same was not the case for Road Cards A and B. G3 employed a similar approach but here, as throughout their construction of solutions, considering each element from the perspective of action or function with no comparison to numbers.

C1: **What about (Road Cards) A and B?**
C2: **If we're smart, we'll refer to the chart. I think it's yes** (closed) **because nothing is added. A is a constant, a preservant, neutral.**
C1: (Referring to the chart) **Look B and B is C.**

C2 then proceeded to list symbollically each possible pair of elements and result for the subset and the three agreed that the subset could not be closed.

In constructing solutions to the sections of each task dealing with **identity and inverse elements and with commutativity**, three of the groups referred to properties of operations with numbers to explain different solutions. G1 discussed whether order mattered for the dolls:

C1: **This is like the commutative property - OBT and OGT or OGT and OBT both equal Both Turn.**
C2: **It's like "Please, my dear Aunt Sally" -addition is commutative. Parentheses, multiplication, division, addition, subtraction -- Wait. Division doesn't work.**
C1: **Addition and subtraction - I don't think subtraction will work either. It's addition and multiplication.**

TABLE 1: GROUP PROBLEM-SOLVING BEHAVIORS

GROUP	TASK	CONSTRUCTION			SOLUTIONS			GENERALIZATIONS		
		Immediate	Corrected	Partial	Concrete	Imagistic	Conceptual	Numbers	Geometry	Problems
G1	Dolls	yes	no	no	yes	yes	yes	yes	no	yes
	Cards	no	yes	no	yes	yes	yes	yes	no	yes
	Roads	no	yes	no	yes	yes	no	yes	no	yes
G2	Dolls	yes	no	no	yes	yes	yes	yes	no	yes
	Cards	yes	no	no	yes	yes	yes	no	yes	no
	Roads	no	no	yes	yes	yes	no	yes	yes	yes
G3	Dolls	yes	no	no	yes	no	no	no	yes	no
	Cards	no	no	yes	yes	yes	no	no	yes	no
	Roads	no	yes	no	yes	yes	no	no	yes	no
G4	Dolls	yes	no	no	yes	yes	yes	yes	no	yes
	Cards	yes	no	no	yes	yes	yes	yes	yes	yes
	Roads	no	yes	no	yes	yes	yes	yes	no	yes

Throughout the five days as they constructed solutions to each of the three problems all of the children were actively engaged in solving the problems although focus varied for individuals throughout the five days. Leadership in Groups 1 and 2 was shared evenly among the members with each individual assuming at different times the more assertive role. In Group 4 one child was noticeably quieter during the first task with the other two both quite assertive. This child, however, was more

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vocal during the second activity and was central to the discussion on the Road Card problem, showing the others that the chart was key to understanding the identity and inverse relationships. Interactions among the members of Group 3 included frequent challenges about strategies and solutions between the two boys, with the girl contributing often but expressing frustration at the boy's arguments. She often attempted to moderate the discussion by offering a statement of consensus. One instance occurred as the children were completing the chart for the Road Cards:

C1: **Any card with Card A will be the same.**
C2: **I suppose that D to D will be A.**
C3: **No - D to B is A.**
C2: **How did you know?**
C3: **Because B to D is A - It's a pattern.**
C2: **You can't prove it. Can you give me a reason for the pattern? Unless you have a reason, we can't just take the hypothesis and must test each one.**
C1: (the girl) **We can say there seems to be a pattern - we have a hypothesis and then we experiment.**

This discussion with both approaches to solution was continued in attempting to determine closure for the four Road Cards:

C3: **It works because of the pattern.**
C2: **No - It's because every possibility is included in these cards.**

The discussion concluded with the explanation given by C2 accepted as the reason for the group's solution.

TABLE 2: ROLE OF INDIVIDUAL WITHIN GROUP

GROUP	CHILD	DOLLS TASK						CARDS TASK						ROADS TASK					
		E	I	A	L	F	M	E	I	A	L	F	M	E	I	A	L	F	M
G1	C1 (boy)	x	x	x	x	x	x	x	x	x	x	x	x	x	x	o	x	x	x
	C2 (boy)	x	x	x	x	x	o	x	x	x	x	x	o	x	x	o	x	x	o
G2	C1 (boy)	x	x	o	x	x	o	x	x	o	x	x	x	x	x	o	x	x	x
	C2 (boy)	x	x	o	x	x	o	x	x	o	x	x	x	x	x	o	x	x	x
	C3 (girl)	x	x	o	x	x	x	x	x	o	x	x	x	x	x	o	x	x	x
G3	C1 (girl)	x	x	o	o	x	x	x	x	x	o	x	x	x	x	x	o	x	x
	C2 (boy)	x	x	x	x	o	o	x	x	x	x	o	o	x	x	x	x	o	o
	C3 (boy)	x	x	o	o	x	o	x	x	x	o	x	o	x	x	x	o	x	o
G4	C1 (boy)	x	x	o	x	o	o	x	x	o	x	o	o	x	x	o	x	o	o
	C2 (girl)	x	x	o	o	x	o	x	x	o	o	x	x	x	x	o	x	o	x
	C3 (girl)	x	x	o	x	o	o	x	x	x	x	o	o	x	x	x	x	o	o

Note: E = engaged, I = interacted, A = (worked) alone, L = led, F = followed, M = moderated (group interaction), x = presence of behavior, o = absence of behavior

Conclusions

Although a variety of the strategies noted in the Tables were used by each group and individuals within the groups to construct solutions, results of the analysis indicated that particular approaches became dominant in the problem-solving behavior of each group. G1 made continual reference to properties of numbers, G2 and G4 based their conclusions primarily on recognized patterns, and G3 referred frequently and almost exclusively to properties and transformation s of physical objects. This group was unique in discussing each operation in terms of action. The analysis also indicated that the children, even as they employed a variety of representations in constructing solutions, did use the concrete representation when questions of meaning arose. The particular dependence on the Road Cards by each of the groups can perhaps be explained because that operation was more difficult to figure out concretely and also because predictions made by each group based on the results of the Klein group structure of the **Dolls Task** proved to be false.

Examination of the data of groups of children working in regular classroom settings suggest that the activities provided opportunities for the children to build cognitive structures by their actions on the objects and their development of connections among representations of the mathematical ideas.

The variety of approaches and strategies employed by individual children and groups suggests that the tasks did provide environments in which the understanding of each child could be developed through the group interaction.

These learning activities carried out as a part of regular classroom instruction without direct teaching indicate the possibility of children working together to construct multiple representations of this particular set of mathematical ideas. One consideration for those committed to constructivist approaches to learning and teaching mathematics is to develop similar problem tasks for a variety of concepts and a range of cognitive abilities. Determining their overall effectiveness as a regular part of instruction is an equally important objective for those seriously contending with the challenge posed by the current national concern about children's mathematical development.

1. For a complete description of the problem tasks, see Alston and Maher (1988).

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CHILDREN'S METACOGNITIVE KNOWLEDGE ABOUT MATHEMATICS AND MATHEMATICAL PROBLEM SOLVING

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This is the first exploratory study in a longitudinal research project involving 20 children (10 boys, 10 girls) randomly selected from a New York City middle school. Its purpose was to examine children's beliefs about mathematics and mathematical problem solving. Data were collected using an interview technique. Procedures for coding the data and establishing the reliability of the coding of the data were designed by the researchers. The results for each interview item were categorized in terms of person, strategy, and task variables. The children's responses suggest that having a good memory, perseverance, and knowledge of mathematics and studying are important for problem solving. However, they seem to lack confidence in their perception of their mathematical ability. They also stated that they check their work, they think it is a good idea to use alternative methods, and they focus on computation involved in a problem. Furthermore, they indicated that the topic of a problem has no bearing on their perception of whether they could solve it, but they felt the length of a problem could affect their ability to solve it. The concept of a "stream of beliefs" and implications for future research are discussed.

During the past decade, researchers in mahtematics education have expressed an interest in many aspects of mathematical problem solving. In particular, articles have focused on various types of mathematics problems (Frederickson, 1984), understanding cognitive proceses and affective components involved in successful probjem solving (Buchanan, 1987; Schoenfeld, 1985), and examining differences between successful and unsuccessful problem solvers (DeFranco, 1987).

One aspect of successful problem solving is reflective thinking which may facilitate monitoring adn checking during a solution process (Kilpatrick 1985). In recent years this phenomenon has been referred to as metacognition.

CONCEPTUAL FRAMEWORK

John Flavell, a cognitive psychologist, has contributed to the development of the study of metacognition. According to him,

> 'Metacognition' refers to one's knowledge concerning one's own cognitive processes and products or anything related to them...Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or object. (1976, p. 232).

He developed a model of cognitive monitoring which can be applied to a variety of cognitive enterprises. Although there are four aspects of the model, this study focused on one of them, metacognitive knowledge. Metacognitive knowledge is similar to knowledge stored in long-term memory. It consists of beliefs about person, strategy, and task variables and how they interact to guide or influence an intellectual endeavor (Flavell, 1979).

Person variables are everything a person comes to realize or believe about himself/herself in relation to his/her own cognitive processes and the problem at hand. Strategy variables consist of approaches to be taken in attempts to resolve the problem. Task variables consist of any information in the problem which may activate beliefs about particular strategies or demands of the problem. In general, these variables can guide or influence an individual to select, change, abandon, or pursue various strategies throughout an entire solution process (Flavell, 1979).

Myers and Paris (1978) and Kruetzer, Leonard, and Flavell (1975), employing an interview technique, studied children's awareness of person, strategy, and task variables in relation to reading and memory performance, respectively. Based on these results and the recommendations of other researchers (Buchanan, 1987; Frank, 1988; Lester & Garofalo, 1987; Schoenfeld, 1985), there is a need to explore children's beliefs about mathematical problem solving.

Research Question

What are fifth graders' beliefs about mathematics and mathematical problem solving in relation to person, strategy, and task variables?

METHOD

Subjects

TWenty fifth graders (10 boys, 10 girls) from a New York City middle school were randomly selected to participate in the study during fall, 1986. The sample reflects the distribution of the school population with respect to ability: 25% below average, 50% average, and 25% above average, determined by standardized tests.

Instrument

An open-ended questionnaire was designed from the work of Myers and Paris (1978). Based on the results of a pilot study, the questionnaire was modified and revised. Eleven questions were designed to measure person variables, 4 were for strategy variables, and 6 were for task variables. The questionnaire was administered individually in an interview by a trained graduate research assistant, who also transcribed the audiotapes.

Data Analysis

For each question from the transcripts, responses were translated into semantically-equivalent summaries (Myers & Paris, 1978). They were then categorized according to variables. The reliability of the tape transcriptions and the coding of the data was established by the researchers.

SOME RESULTS

For the person variable questions, the children's responses suggest that having a good memory, perseverance, knowledge of mathematics, and studying are important for success in problem solving. However, they seem to lack confidence in their perception of their mathematical ability.

For the strategy variable questions, they indicated that they check their work, they think it is a good idea to use alternative methods to solve a problem, and they focus on the computation involved in solving a problem. It seems as though "checking" is limited to reviewing the computation in the algorithm selected and it does not include monitoring or checking for the reasonableness of an answer.

For the task variable questions, the children indicated that the topic of a problem has no bearing on their perception of whether they could solve it, but they felt that the length of a problem could affect their ability to solve it.

Stream of Beliefs

As the results were being examined, it seemed natural to discuss the responses to some questions together. The term "stream of beliefs" is used to identify a relationship between or among beliefs when it seems as though one or more beliefs affect other beliefs.

In figure 1, there are 3 children's responses to 2 person variable questions. The responses in Protocol A reflect a complementary stream of beliefs because each response supports the other. The responses in Protocol B reflect a conflicting stream of beliefs because although she trusts her memory for obtaining correct answers, she does not feel it is important to have a good memory to solve a problem. Although the responses in Protocol C seem to reflect a complementary stream of beliefs, if he solely depends on his memory during problem solving, a failure to recall necessary facts may lead to a lack of perseverance and failure.

In figure 2, there are 2 children's responses to 2 strategy variable questions. The responses in Protocol D reflect a complementary stream of beliefs because her comments support her belief about checking with respect to the reasonableness of an answer. The responses in Protocol E reflect a conflicting stream of beliefs because she admits to checking her work for the purpose of getting the correct answer, but she is willing to accept an unreasonable answer based on correct computation.

FINAL REMARKS

This study was exploratory and descriptive. Its purpose was to examine fifth graders' beliefs about mathematics and mathematical problem solving in relatio to person, strategy, and task variables.

Since this qualitative study was dependent upon children's ability to express themselves verbally, some information may have been inadvertently omitted or lost. However, we believe that by interviewing children we obtained a rich source of accurate data.

The results of this exploratory study support the need for further research. In particular, future studies should include the examination of the relationship between children's beliefs and their problem-solving behavior. Different aspects of person, strategy, and task variables should be examined to study the relationship between beliefs and behavior in a problem-solving setting. Also, longitudinal studies should be conducted at various ages and grade levels to examine how cognitive as well as non-cognitive factors influence individual belief systems and mathematical performance.

1. Do you think it is important or not so important to have a good memory in order to solve a math problem? Why?

Protocol A[a]	Protocol B	Protocol C
It's important. Because if let's say ...if somebody didn't have a memory ...they couldn't answer anything. They couldn't talk. They couldn't read. They couldn't solve anything ...if they had a memory they could solve everything like talk intelligently, solve problems, solve multiplication problems for math, many more.	Well I don't think you have to have a good memory because you could just learn it that year or that minute and know what you're doing.	I think it's very important. Because if you don't remember or memorize how to solve a math problem you won't be able to solve it.

2. The other day I met a fifth grader who said it's important to remember certain facts to do well in math. When you take a math test, do you or do you not trust your memory to remember important facts? Why?

Protocol A	Protocol B	Protocol C
I trust my memory. Since my memory takes in information, and, if I read my math–like my math test–it's like having a cabinet in my brain with all these cards with mathematics signs, with mathematics answers, Anything I studied.	I do trust my memory...because my memory...it could be the right answer and I could get a lot of help from that.	I trust my memory, most of it. Because...it's easy when you remember the problems as like division and you have to divide, multiply and like that.

Figure 1. Sample responses to Person Variables, Questions 1 and 2.

3. When you solve a math problem, do you or do you not check your work? Why/Why not?

Protocol D[a]	Protocol E
I check my work. Because, I, um, try and get my examples right as much as I can so that when I give them to the teacher, the teachers knows that I am trying my best and that I'm not just putting any answer down.	I check it. Because to see if I got it right and if I got wrong, I'll do it again and see which one is right.

4. A 5th grader was given this problem to solve: 128 children are going on a school trip. A bus can seat 40 children. How many buses are needed so that all the children can go on the trip? His/Her answer was 3 1/5 buses. Would you give this same answer? Why/Why not? (The following work was shown to each child:

$$
\begin{array}{r}
3\,8/40 = 3\,1/5 \\
40 \,\overline{)\,128} \\
\underline{120} \\
8
\end{array}
$$

Answer: 3 1/5 buses

Protocol D	Protocol E
I would just put down 4 buses because you can't really give 3 1/5 buses, so 4 buses is the closest. Because you can't have 3 buses 'cause then you'd have more children. But if you have 4 buses then all the rest of the children can go on another bus.	Well it might be wrong because ...wouldn't it be lie 3?... no it could be right because 3 remainder 8 might be right.

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[a] Protocols labeled with the same letter represent the same subject.

Figure 2. Sample responses to Strategy Variables, Questions 3 and 4.

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BUILDING A QUALITATIVE PERSPECTIVE BEFORE FORMALIZING PROCEDURES: GRAPHICAL REPRESENTATIONS AS A FOUNDATION FOR TRIGONOMETRIC IDENTITIES

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This study compared two approaches to incorporating graphical representations into a unit on trigonometric identities. The first treatment *supplemented* a traditional approach with related graphing activities. The second treatment used graphical representations as the *foundation* for trigonometric identities. Subjects in the second treatment showed superior posttest performance in relating functions to graphical representations, as well as more variety and personal involvement in their approaches to the standard content.

Trigonometric identities are traditionally approached as exercises in symbol manipulation. The instructional methods used in this study combined the usual symbol manipulation work with guided activities using graphical representations. Two conditions were compared:

1. A traditional approach to trigonometric identities, *supplemented* with related graphing activities, and
2. A graphical approach, using graphical representations as a *foundation* for trigonometric identities.

The study was imbedded in a normal classroom instructional sequence for trigonometry. The subjects, 30 students in grades ten through twelve, had completed the introductory trigonometry material from their textbook (Dolciani & Wooton, 1980, Chapter 14) and were ready to begin identities. The textbook material had been augmented with computer-related activities designed to introduce the graphs of the six basic trigonometric functions, provide experience with transformations of sine and cosine graphs, and establish technical facility with the graphing software to be used in the experiment.

A test was given to summarize the introductory material and assess achievement on the first chapter of trigonometry. Following this chapter test the subjects were divided randomly into two treatment groups. One group was assigned the Supplemented Traditional (ST) approach; the other, the Graphical Foundation (GF) approach.

Both treatments consisted of five days of class sessions, microcomputer laboratory activities, and homework. Microcomputer activities for both treatment groups were guided by activity sheets designed by the experimenter. During the microcomputer activities, subjects in each treatment group worked with partners of their choice, in order to encourage verbalization and facilitate exchange of ideas and insights.

All subjects used acetate slides and overhead projector pens to draw graphs on the microcomputer screens, as required by the activity sheets. In general, ST subjects *recorded* graphs that had been plotted by the computer, whereas GF subjects *predicted* graphs and used the computer to check their predictions. The overall characteristics of the treatments are summarized in Table 1. A more complete description is contained in Dugdale (in press).

Table 1

Comparison of Characteristics of Treatments

Supplemented Traditional (ST)	**Graphical Foundation (GF)**
Trigonometric identities were treated in the traditional fashion, as exercises in symbol manipulation. Graphs were used as an additional representation.	Trigonometric identities were introduced graphically, and the usual symbol manipulations were used to justify the relationships evidenced in the graphing activities.
Symbol manipulation exercises were preceded by a straightforward presentation of the eight fundamental identities and direct instruction covering procedures to be used.	Subjects were asked to justify algebraically the equivalence of functions without being instructed how it should be done. Part of the task was to decide what information was applicable and how to use it.
Computer activity sheets were arranged with a worked-out example preceding each set of exercises. Exercises were routine repetitions of the procedure used in the example.	Computer activity sheets presented non-routine tasks, some of which required analyzing graphic feedback and revising functions to change their graphs.
Guidance provided on activity sheets focused on what procedures subjects should apply.	Guidance provided on activity sheets focused on what questions subjects should address.
Relationships between graphs (such as the correspondence between the zeros of a function and the asymptotes of its reciprocal function) were presented, but the activities did not require subjects to use these ideas.	Subjects were asked to use graphs of functions to predict graphically the shapes of other functions before plotting. For example, from graphs of $y=\sin x$ and $y=\cos x$, subjects figured out where $y=\sin x/\cos x$ would have zeros and asymptotes and predicted the general shape.

RESULTS

A posttest included two sections. The Proving Identities Section asked subjects to prove two given identities, typical of those in the textbook exercises. The Graphical Representations Section was a multiple-choice test requiring subjects to choose an appropriate graph for each of eighteen given functions.

Using the previous chapter test as the covariate, analysis of covariance was used to compare subjects' scores on each section of the posttest, and also on three categories of items within the Graphical Representations Section. Results are presented in Table 2.

Table 2

Results of Analysis of Covariance

Item Type	Treatment[a]	Mean (SD)	Adjusted Mean	$F_{1,26}$	p
Proving identities					
Total Score	ST	8.714 (1.899)	8.393	1.408	.246
	GF	7.188 (3.082)	7.430		
Graphical representations					
Total Score	ST	21.464 (8.763)	19.870	7.696	.010
	GF	25.188 (7.101)	26.012		
Basic Material	ST	10.321 (4.107)	9.577	5.197	.031
	GF	11.281 (2.738)	11.656		
Covered, Not Basic	ST	8.214 (3.720)	7.753	3.431	.075
	GF	9.313 (2.323)	9.576		
Non-routine Items	ST	2.928 (2.433)	2.540	5.419	.028
	GF	4.594 (3.018)	4.780		

[a] For ST, n=14; for GF, n=16.

For the standard content of proving trigonometric identities, the data in Table 2 indicate no significant difference (p=.246, α =.05). (Scores for the Supplemented Traditional Treatment were somewhat higher than for the Graphical Foundation Treatment on this section of the posttest, but the difference was not statistically significant.)

GF subjects showed significantly higher (p=.010) posttest performance on relating trigonometric functions to their graphical representations. Further, within the graphical representations section of the posttest, GF subjects showed:

1. Significantly higher (p=.031) performance on items requiring use of basic material. These 7 items involved recognition of the graphs of basic trigonometric functions and simplification of trigonometric expressions by routine symbol manipulation.
2. Higher (though not significantly, p=.075) performance on items requiring use of more advanced material covered by both groups. These 6 items involved graphs of squares of basic functions and transformations of sine and cosine functions.
3. Significantly higher (p=.028) performance on non-routine items. These 5 items involved functions which would not conveniently reduce by symbol manipulation to basic functions or squares of basic functions and would not analyze readily as transformations of sine or cosine functions.

DISCUSSION

Proving Trigonometric Identities

In approaching the symbol manipulation content, the treatments differed in their handling of prerequisite knowledge and procedures. The ST Treatment began with a presentation of the eight fundamental identities, thereby sorting out for the subjects what particular subset of their knowledge they were to use. The GF Treatment included no introductory presentation of the content to be used, leaving the subjects to consider what of their previous knowledge could be applicable. Five of the eight fundamental identities had been introduced in the previous chapter in the context of defining the trigonometric functions and expressing the relationship between the sine and cosine functions. Subjects were already familiar with the following five identities and their restrictions:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \qquad \sec\alpha = \frac{1}{\cos\alpha}$$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} \qquad \csc\alpha = \frac{1}{\sin\alpha}$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

Subjects were not familiar with the following three identities:

$$\tan\alpha = \frac{1}{\cot\alpha}$$

$$\tan^2\alpha + 1 = \sec^2\alpha \qquad 1 + \cot^2\alpha = \csc^2\alpha$$

These last three identities are convenient, but unnecessary, because they derive easily from the first five. Hence, the GF Treatment was *not* a case of putting students into a situation for which they were unprepared and asking them to discover or develop whatever they needed. Rather, it was a case of asking students to apply what they *already knew* from another context, without telling them what *in particular* to apply.

The ST Treatment included a presentation of each necessary procedure prior to its use, so that the subjects' goal was to apply the given procedure to the given items. The GF Treatment included no initial presentation of procedures for proving identities, so that the subjects' goal was to devise a convincing argument for an observed equivalence. GF subjects were asked to justify identities without *procedures* given, but not without *context*. In justifying what they had observed graphically to be true, subjects were expected to understand the goal in a qualitative sense, and, hence, have a good chance of success in formalizing their knowledge.

The initial symbol manipulation work of ST subjects was generally cleaner and more standard than that of GS subjects. GF subjects approached the task with more variety in their methods. For example, one GF subject decided to justify the observed equivalences by using definitions of trigonometric functions in terms of a right triangle with sides a, b, c, using $\sin x = a/c$, $\cos x = b/c$, etc., although the two classmates with whom she was working chose a more standard approach. By the end of the second class session, this subject had verified to herself that her method was essentially equivalent to what her classmates were doing and that their approach was probably less cumbersome. Although she abandoned her initial method, it provided some synthesis between the current topic and earlier material, and the ownership she felt for her method was clearly important to her. Another subject initially used the right triangle definitions, $\sin x = \text{opp/hyp}$, $\cos x = \text{adj/hyp}$, etc., then changed to the method her classmates were using.

It was not expected that the GF Treatment would be more effective than the ST Treatment in building subjects' skills with the standard content of proving identities. A loss of instructional efficiency would not have been surprising. However, it was anticipated that whatever loss of performance might be evidenced in the standard content of proving identities, it would be outweighed by improved performance in relating trigonometric functions to their graphical representations.

Relating Functions to Graphical Representations

ST subjects were exposed to more graphical representations than were GF subjects (though the GF subjects did better on this section of the posttest). For example, in drawing graphs on their acetate slides, ST subjects drew the graphs of 18 different equations, 6 of which were on the posttest. In contrast, GF subjects drew the graphs of only 7 equations, one of which was on the posttest. ST subjects did routine work with many graphs, while GF subjects were involved in more thoughtful work with fewer graphs.

There was a noticeable difference in the graphs drawn by the two groups. ST subjects tended to produce more uniformly neat, accurate, and properly labelled graphs on their acetate slides. In contrast, GF subjects were more likely to produce sketchy, sometimes incomplete, graphs. ST subjects appeared to regard their work as a finished product, while GF subjects approached the task more as scratch work on the way to a solution.

GF subjects' inaccuracies in predicting some graphs became obvious when these subjects checked their work by having the computer plot the graph. This raised the question of *why* a graph would differ from the prediction. ST subjects, recording computer-plotted graphs, lacked opportunities to make errors that would raise questions. Given a graph that was rounded, ST subjects were not likely to question why it was not pointed. However, for GF subjects, first predicting the shape, then checking it, raised interest in whatever features turned out to be different from what had been predicted.

Given that GF subjects had learned to predict the shape of a graph before plotting, it is not surprising that they performed better than ST subjects on posttest items that did not conveniently reduce by symbol manipulation to easily recognized graphs and that also did not analyze readily as transformations of sine or cosine graphs. GF subjects had more experience with an additional method of approaching graphs, although, in fact, five of the seven graphs that GF subjects were asked to predict during the treatment would have been easily (and more efficiently) accessible by symbol manipulation into basic graphs or squares of basic graphs.

The GF Treatment's emphasis on graphically predicting the shape of a function may account for the difference in posttest performance on the non-routine items, but it does not explain the difference for other categories of items, particularly those requiring only recognition of basic graphs and use of routine symbol manipulation.

CONCLUSION

In addition to using graphical representations as the foundation for trigonometric identitites, the Graphical Foundations Treatment was intended to involve students in:

1. Experiencing active participation in the development of mathematical ideas. Students were to predict and figure out, rather than follow examples, copy graphs, and have ideas explained.
2. Building a qualitative perspective before formalizing procedures. Trigonometric identities were introduced graphically, and the usual symbol manipulations were used to justify algebraically the relationships observed in the graphing activities.
3. Applying previous knowledge and skills to a current problem without being told what, in particular, to do. Students were to decide what of their previous knowledge was applicable and devise convincing arguments for observed equivalences. Students were involved in learning more generally-applicable inquiry techniques in addition to basic content.

The results of this study suggest that a graphical approach, with careful attention to students' experiences beyond the immediate content goals, can produce a richer learning experience without significant detrimental effect on the mastery of standard content. In addition to showing superior posttest performance in relating functions to their graphical representations, subjects in the Graphical Foundations Treatment exhibited more variety and personal involvement in their approaches to the standard content.

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Area and Perimeter, Applied Problem Solving and Constructivism

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"We need to be careful not to put the constructivist cart before the values horse" (Kilpatrick, 1987, p.21).

A theory of knowing can never determine for us an answer to the question of what it is we should come to know; i.e., function cannot define value. On the other hand, it may be that values can influence function. Explicit values, expressed in a learning context evident to the learner, can help give rise to the purposeful inquiry of the learner, and thereby enhance whatever are the functioning mechanisms by which we learn. In the research that we report here, explicit *social* values helped create just such a context for learning. Within that context, students purposeful inquiry while engaged in applied problem solving, contributed to their learning of area and perimeter concepts.

Perspective and Background

In this research on the teaching and learning of area and perimeter concepts, our purposes rested on the value judgment that for any student, knowledge of foundational geometric ideas is clearly a preferred state. Arguments for the study of geometric concepts are various, but in particular, even an elementary understanding of area and perimeter offers a vast potential as a basis with which learners may form and generalize their understanding of many mathematical and scientific ideas. Some of these ideas include certain obvious cases, such as those found in measurement concepts in the extension from linear to multidimensional settings. Perhaps less evident cases relate to a generalized perimeter or "boundary" concept; one necessary in the simple dichotomous logic arising in two variable linear inequalities, in the notion of a line integral, in arc length, in boundary value problems, in the Jordan curve theorem, in the open and closed sets of topology, in the contour integrals of complex analysis and in a vast number of other cases.

However, in our research, as will nearly always be the case, students were not and could not have been aware of any of our deeper educational aims; our lofty notions, the "good" of our program. As a consequence, in our instructional model, we tried to directly address the problematic question of developing and sustaining, throughout the teaching program, what is the interest, or more essentially, the intellectual curiosity of students.

Since we believe in a conception of mathematics as a highly developed superstructure of interwoven concepts which nevertheless rests on a base of historical necessity and practical consideration, our concern was that students might *experience* some small measure of that concept for themselves. As a result of that experience, we believed that students might then be more likely to believe, in the sense of Thom (1973), in the *existence* of that which we call mathematics.

In an attempt to overtly demonstrate that mathematics is related to actions on the world, our model chose a social context in which to develop the teaching program; one that, by our judgment, would be relevant to the 67 adult subjects of the study. That social context was the urban redevelopment of Newark, N. J.; the city in which most of the students lived. Our sense of adults in a remedial mathematics course was that they needed to be moved away from a conception of mathematics as limited to topic review (however thorough) and toward what might be a mathematics of *relevant* problem solving. Part of our job as educators was to try to provide a setting for the directed program experiences of students so that they might continually redefine their notion of "relevance" and ultimately decide that with respect to certain specific questions, it was indeed exactly mathematical concepts that were most relevant. Within the context of a city's *renaissance*, our model established a potentially rich setting for applied problems. Our instruction was flexibly but firmly directed toward the encouragement of student questions and discussions that were almost inexorably predisposed toward aspects and issues of form and quantity; i.e., mathematical concepts.

Applied problems provided the vehicles in our model by which student understanding was transformed. But, first we had to create a context in which subsequent classroom activity, *from the learner's perspective*, would seem purposeful. All of the utilized problems are described in detailed classroom protocols in the completed research (Pace, 1989a). Previously described in some detail (Pace, 1989b; Pace & Maher, 1989) is "the shopping mall problem". Briefly, this initial problem posed to small groups of students how they might choose a "best" parcel of land for a joint business venture in a hypothetical, newly constructed Newark shopping mall. A process of open-ended discussions lead to a variety of questions ranging somewhat unpredictably over concepts found in real estate, marketing, tax law, mortgage loans, etc..., and including those of area and perimeter, for which our program was designed. With the guidance of their instructor, students were able to extract mathematical concepts from a lifelike situation rather than what might be a more typical converse situation; i. e., where students are taught a mathematical concept and then must try to somehow imagine where, how or why any such idea might arise.

Results with a second set of problems have also been previously reported (Beattys & Pace, 1988). This second set of applied problems developed measurement tasks through activities concerned with the covering and framing of posters. Another activity, gave the students a representational drawing of a square foot and asked for a drawing representative of a square yard.

Selected Quantitative Results

The research design, as previously detailed (Pace, 1989a, 1989b), enabled all subjects to act as part of both control and experimental groups. At 2 1/2 week intervals, subjects were pretested, posttested, and tested for retention (delayed test) with three forms of the "Applied Geometry Test" (AGT) (available from the author upon request). This 19 item test combined a few elementary items from the "Van Hiele Geometry Test" (Usiskin, 1982) and included a somewhat larger number of applied area and perimeter problems. Scoring was done by two experienced mathematics teachers utilizing strict grading criteria for partial credit.

Out of a possible score of 100, the subjects' mean AGT pretest score was 36.5. Referring to the density function of Figure 1, we note that 86.6% (58/67) scored less than 50%.

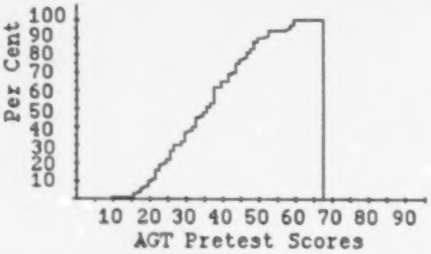

Figure 1. Cumulative density function for Applied Geometry Test pretest for all subjects.

Through the use of a sequence of stepwise linear and multivariate regression models, all of whose parameters were estimated by the computer program, Regress II (Madigan & Lawrence, 1983), we found a significant increase in performance and retention on the AGT.

Posttest Prediction

Regression model 1, which predicted AGT posttest score as a function of membership in the experimental or control groups and AGT pretest score, yielded the following equation:

$$y_p = 4.216 + 29.169x_0 + 0.895x_1 \qquad (1)$$

Equation 1 indicates that posttest prediction score is a function of two variables; x_0, membership in the experimental or control group and x_1, AGT pretest score.

Since $x_0 = 1$ for experimental group and 0 otherwise,

$y_p = 33.386 + 0.895x_1$, for the experimental group, and

$y_p = 4.216 + 0.895x_1$, for the control group.

These equations are plotted in Figure 2. The results of stepwise linear regression model 1 indicate conclusively that membership in the experimental group is a significant factor in AGT posttest performance. The approximate average difference of 29 points between students of similar AGT pretest achievement in the experimental and control groups, we attribute to the experimental program of teaching.

Equally as significant as the above result was the fact that a multivariate regression model, incorporating "van Hiele levels" and scores from the New Jersey College Basic Skills Placement Test (NJCBSPT) as additional predictor variables, could offer no better prediction equation than that of equation 1.

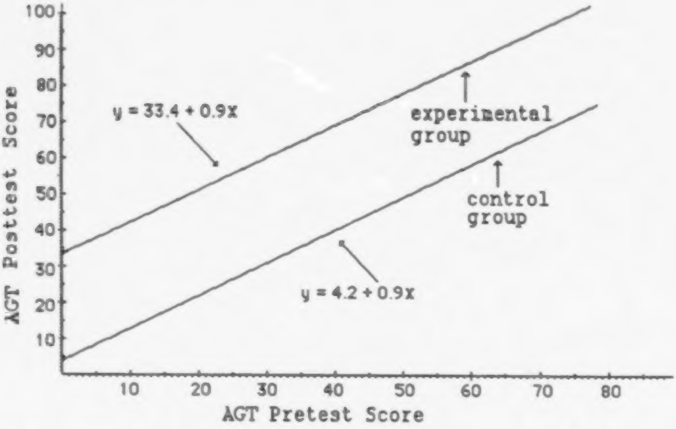

Figure 2. Predicted Applied Geometry Test posttest scores as a function of Applied Geometry Test pretest scores.

Retention

A general multivariate model for predicting "delayed" AGT posttest achievement considered AGT pretest, AGT posttest, van Hiele level and NJCBSPT score as predictor variables. The regression model yielded the following prediction equation:

$$y_d = -2.6698 + 0.4282x_1 + 0.7176x_5 \qquad (2)$$

Equation (2) indicates that y_d, predicted scores on the AGT delayed posttest are a linear function of scores on AGT pretest, x_1 and posttest, x_5; i.e., $y_d = f(x_1, x_5)$. This function is represented by the plane graphed in Figure 3.

To offer a simple interpretation of the graph of Figure 3, consider two students, one with (x_1, x_5) AGT pretest/posttest scores, and the second with $(x_1+\partial_1, x_5+\partial_5)$ AGT pretest/posttest scores. The difference, then, in expected delayed posttest achievement is:

$$y_d(x_1+\partial_1, x_5+\partial_5) - y_d(x_1, x_5) = 0.4\partial_1 + 0.7\partial_5$$

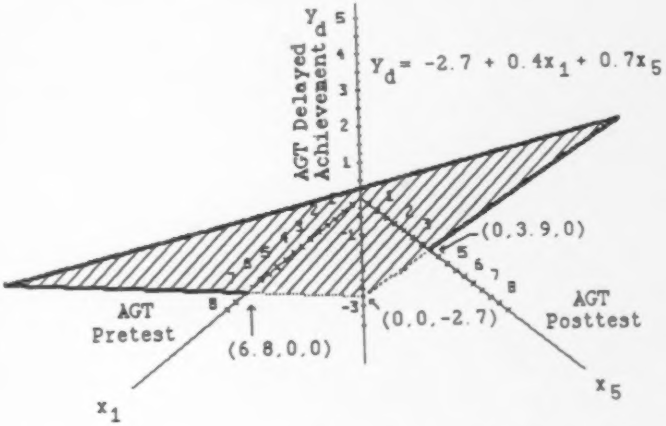

Figure 3. Applied Geometry Test delayed achievement as a linear function of the two variables; Applied Geometry Test pretest and Applied Geometry Test posttest (Isometric View-Planer Region).

Summing Up

When taught area and perimeter concepts through an unconventional program of applied problem solving developed within a social context designed to interest adults, students demonstrated dramatic and retained achievement. Could they have learned these concepts without such an approach? At some reasonable level, we are certain that they could have. But our argument is as much for the process of education as it is for any measurable result. Long after "facts" are forgotten, the process; a construction, lingers on. It is just this kind of construction, or perhaps more appropriately, this kind of reconstruction, in a purposeful direction, that might provide the "stuff" of which mathematics education could be made in a different way.

Whether our epistemological "cart" be of constructivist design or otherwise, it is ultimately the values of the "driver" (and not the fancy of the horse) that will determine both our direction and destination.

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PROBLEM POSING BY MIDDLE SCHOOL MATHEMATICS TEACHERS

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The problem posing and conjecturing of middle school mathematics teachers was examined by asking them to produce conjectures in a task environment that allowed exploration of a rich variety of mathematical relationships. Subjects generated conjectures both before and after attempting to solve a specific problem embedded in the same task environment. The findings suggest that the teachers could generate reasonable, interpretable conjectures and problems related to many aspects of the task environment and that there were qualitative differences between the conjectures generated before solving the specific problem and those generated afterwards.

Progress has been made in the past two decades in understanding many aspects of mathematical problem solving (Fredericksen, 1984; Silver, 1985). Research on problem solving, however, has focused almost entirely on problems which have been formulated in advance for the solver. Although problem posing and conjecture were at the heart of George Polya's writings on problem solving (1954, 1957, 1981), and although Polya's work has spawned much of the interest in mathematical problem solving as a field for research in mathematics education, very little research attention has been focused on the important processes involved when solvers generate their own problems (Kilpatrick, 1987).

Given the centrality of problem-posing processes not only in the discipline of mathematics and the nature of mathematical thinking but also in current efforts to reform the character of precollege mathematics education, research that deals directly with problem posing is needed. The study reported here examined the problem-posing behaviors of middle school mathematics teachers. Teachers were viewed as important subjects in the study of problem posing because they represent a quasi-expert population whose knowledge of mathematics is likely to exceed that of the students they teach, and because it is important to know if precollege mathematics teachers themselves can and do engage in these generative processes in their own problem solving. Since recent calls for reform of precollege mathematics (e.g., NCTM, 1989) argue that problem posing should be a regular feature of middle school instruction, the implication is that problem posing is accessible not only to all students at these grade levels but also to their teachers. The purpose of this study was to investigate the ability of middle school mathematics teachers to generate and pose interesting mathematical conjectures or problems. Subsidiary purposes were to investigate the adequacy of the task and methodology utilized in the study and the influence, if any, of collaborative pairing on the generative products.

METHOD

Subjects Data were obtained from 53 teachers of middle school mathematics who attended a Summer workshop sponsored by their school district. The mathematics background of the subjects varied from a Bachelor's degree in mathematics to no formal collegiate-level mathematics coursework. The 53 teachers produced 39 samples of work, since 25 worked individually and 28 worked in pairs.

Task A Billiard Ball Mathematics (BBM) task, in which an imaginary billiard ball is projected from the lower left corner of a rectangular table at an angle of 45 degrees to the sides, was used as a task environment in which subjects could generate (and solve) interesting mathematical problems.

Several questions can be asked about the behavior of the ball in the BBM task; each question can be the basis for posing a mathematical problem or generating a conjecture. For example, one can ask, "Will the ball always land in a pocket?"; "Can we predict the final pocket into which it will fall?"; or "Is there a relationship between the dimensions of the table and the final pocket or the number of "hits" made by the ball on the sides?". The BBM task can be viewed as an experimental task domain in which the independent variables are the table's length and width and the gradient of the ball's path. Several interesting dependent variables are the final pocket (into which the ball would fall), the number of "hits" made on the sides by the ball on its path, the number of squares passed through by the ball on its path, and the number of regions formed by the trace of the path of the ball. Determination of relationships between and among dependent and independent variables requires knowledge of elementary number theory, involving only concepts and skills studied routinely in middle school (e.g., factors, multiples, least common multiple).

Task Presentation In this study, three paper and pencil tasks, embedded in the general BBM task environment, were administered to the subjects. The three tasks were administered during a fixed time period. The Initial Conjecturing (IC) task lasted 10 minutes, and it was followed by the Problem Solving (PS) task, which lasted 35 minutes, and the Additional Conjecturing (AC) task which was available during the PS task and during an additional 5 minutes.

The IC task consisted of a brief description of the basic BBM task environment, accompanied by two examples of particular billiard tables and the path of the ball on each table. For each example, the dimensions of the table, the path of the ball, the final pocket and the number of hits on the sides were pointed out, and the subjects were directed to write down any problems or questions they could think of related to this setting. In order to stimulate the broadest possible array of conjectures or problems, directions for the IC task were deliberately nondirective about the sort of questions or problems that should be written by the subjects.

For the PS phase, a particular problem was posed for the subjects. They were instructed to: "Look at the examples, think about the situation for tables of other sizes, consider as many examples as you need, and try to predict the final destination of the ball. That is, when will the ball land in pocket A? When will it land in pocket B?..in pocket C?..in pocket D?" During the PS phase, grid paper was freely distributed to the subjects, since it was expected that their solutions would be based on an empirical approach.

The directions for the AC task repeated those given for the IC task. The placement of the IC and AC tasks before and during/after the PS task was designed to explore how the empirical work on the specific problem might affect the quantity or quality of conjectures generated by the subjects.

RESULTS

In this paper, the responses to the IC and AC tasks are called conjectures, and the responses to the PS task are called solutions. Subjects' conjectures and solutions were classified within a broad, general categorization scheme which was created to be suitable for analyzing a wide range of conjecturing and problem-solving tasks. In our discussion of results, we will first present our analysis of the conjectures produced during the IC and AC phases, then we will present data concerning the solutions produced during the PS phase. Finally, we will present our preliminary analysis of the relationship between subjects' conjecturing and problem solving.

Conjectures Generated in the IC and AC Tasks

The conjectures were divided into four categories which partitioned the set of interpretable responses into mutually exclusive subsets. In order to make the partition exhaustive for the entire set of responses, a fifth category , consisting of uninterpretable responses, was added. A brief description of each category is given in Figure 1. After the categories were developed, each conjecture was coded independently by two raters. Use of these categories resulted in an acceptably high degree of inter-rater reliability (Kappa = 0.91). After computing the reliability, the few discrepancies between the two raters' categorizations were resolved through discussion before the final tallies were recorded for each category.

Table 1 presents a summary of conjectures produced by individuals and pairs on the IC and AC tasks. The data indicate that subjects were more productive in posing conjectures during the IC phase, and that the rate of production during both phases was similar for individuals and for pairs. Subjects were not "pure" in their conjecturing and tended to produce conjectures in more than one

Figure 1 Categories of Conjectures Produced during the IC and AC Tasks

Category 1 (Implicit Assumptions) — Conjectures related to varying the assumptions which are implicit in the BBM task situation, such as introducing spin or collision with other balls; varying speed, force or momentum; or allowing a difference between the angle of reflection and the angle of incidence after a hit.

Category 2 (Initial Conditions) — Conjectures related to varying the initial conditions of the task, such as changing the angle at which the ball is projected or the initial position from which it is propelled.

Category 3 (General Goals) — Conjectures related to general goals, such as questions seeking a relation between the dimensions of the table and either the number of hits on the sides of the table or the number of squares and triangles traced by the ball.

Category 4 (Specific Goals)— Conjectures related to specific goals, such as questions about the final pocket or total number of hits for a table with specified dimensions or the dimensions needed for a specified final pocket to be reached.

Category 5 (Other) — Conjectures which are vague or difficult to understand or which do not appear to be formulated in the form of a question about the task environment given.

Table 1 Distribution of Conjectures by Task and by Category

		Category					
	Phase	1	2	3	4	5	Total
Individual	IC	16	7	18	15	18	74
(N=25)	AC	6	7	10	23	20	66
Pairs	IC	10	6	7	7	20	50
(N=14)	AC	10	3	1	10	10	34
Total		42	23	36	55	68	224

category. Subjects produced more conjectures in Category 4 (related to specific goals) in the AC task than in the IC task. On the other hand, subjects were more likely to produce conjectures in category 3 (related to general goals) in the IC task. These results probably show the influence of the intervening PS task. Overall, conjectures in Category 4 were the most common, accounting for 35% of the interpretable conjectures produced; whereas, conjectures in Category 2 were the least common, accounting for only 15% of the interpretable conjectures.

Solutions Produced in the PS Task.

The responses were divided into four categories which partitioned the set of interpretable solutions or attempted solutions into mutually exclusive subsets. The categories were based primarily on the completeness of the attempted solution and the generality of the approach taken. In order to make the partition exhaustive for the entire set of attempted solutions, a fifth category , consisting of solutions that were misdirected or difficult to understand, was added. A brief description of each category is given in Figure 2.

Figure 2 Categories of Attempted Solutions in the PS Task

Category I (Simple List) — Attempted solutions involving the construction of a simple list consisting of worked examples for particular dimensions.

Category II (Quasi-General) — Attempted solutions which are broader than simple lists, in that some consideration of ratio is evident, or one dimension is fixed while the second dimension is varied.

Category III (General, but Case Bound) — Attempted solutions which are intended to be general but which are confined to consideration of only one particular final pocket.

Category IV (General) — Attempted or complete solutions which are based on consideration of all possible dimensions.

Category 0 (Other) — Attempted solutions which are misdirected or ambiguous.

Table 2 contains a summary of the solutions produced by individuals and pairs. The small number of solutions (only 6 of 39) in the higher categories (III or IV) indicates that most subjects did not produce complete solutions. Particularly striking was the poor performance of the subjects working in pairs on the PS task. It would appear that the solution performance for both individuals and pairs would have been stronger if there had been more time available. Several partial solutions contained promising local results which could have been extended to provide a more general solution if more time were available to the subjects. Subjects operating in pairs might have been especially hampered by the time constraint.

Table 2: Number of Solutions for Individuals and Pairs by Category

	Category					
Responders	0	I	II	III	IV	Total
Individuals	5	9	6	3	2	25
Pairs	6	4	3	0	1	14
Total	11	13	9	3	3	39

Relating Solutions and Conjectures

Possible relationships between the conjecturing and problem-solving behaviors of the subjects were also examined. A few trends were evident, but there appeared to be no direct, simple relationship between the two data sets. For example, during the AC task, subjects who produced "weaker" solutions in the PS task were about three times as likely to make conjectures related to implicit assumptions than subjects producing "stronger" solutions; but there was no similar relationship for the IC task. Subjects who produced the strongest solutions on the PS task also tended to make a larger percentage of their conjectures on the AC task in category 4, but there was no similar trend for the IC task. In fact, on the AC task there was a general trend for all subjects, regardless of their success on the PS task, to make more conjectures in category 4 and fewer in category 1. The tendency of subjects to produce conjectures related to specific goals on the AC task is likely due to the influence of the PS task.

DISCUSSION

The BBM task appeared to function well as a micro-environment for mathematical conjecturing. In particular, the responses to the IC task were rich in variety. The middle school teachers in our sample were able to generate a reasonable number of interpretable conjectures under the conditions of the tasks presented to them. Moreover, the solutions produced in the problem-solving task gave an indication that the teachers were capable of making reasonable progress toward an empirically-based solution to the problem that was posed. Therefore, our evaluation of the task and methodology is cautiously optimistic.

The results also suggested that it is reasonable to examine problem posing and conjecturing both before and during/after problem solving. In fact, the different kinds of conjectures produced by subjects on the IC and AC tasks point to a potentially interesting interaction between their problem-posing and problem-solving behaviors. The specific question given in the PS task probably diverted subjects' attention away from general conjectures and directed it toward specific conjectures similar to those embedded in the PS task. Since post-solution conjecturing is an important component of mathematical thinking (Brown & Walter, 1983), this phenomenon deserves further research attention and analysis.

The tasks and methodology employed in this study appeared to be suitable for gathering, in a relatively short time, a large amount of data on this topic. Several limitations, however, constrain our ability to interpret our findings and make generalizations. For example, the lack of clinical interviews — a natural alternative to our large-group, written tasks — somewhat hampered our analysis and interpretation of responses. It is difficult to know whether the fairly large number of uninterpretable responses was due to a basic misunderstanding of the task requirements, a difficulty in communicating conjectures and problems, or subjects' mathematical weaknesses. Nevertheless, the group-administered, paper-and-pencil BBM tasks appear to be viable as components of an assessment scheme designed to study or evaluate mathematical problem posing.

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Factor Structure of Junior High School Students' Responses to Metacognitive Statements for a Non-routine Problem

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Abstract

A series of 21 statements were designed to elicit student awareness of cognitive activities when working on a non-routine mathematical word problem. Four hundred and twenty-one junior high school students responded to these metacognitive statements as part of a larger study of student's thoughts and feelings about mathematical word problems. Based upon a principal axis factor analysis, clusters of items were identified. These were student awareness of processing activities before, during and after solving the non-routine problem, as well as general problem solving activities. The results suggest a paper and pencil measure may help teachers and researchers to assess student awareness of cognitive activity during problem solving.

Objective

The objective of this study was to examine the factor structure of a set of statements that were intended to elicit students' awareness of their active monitoring and regulation of their cognitive processes during the solution of a specific non-routine mathematical word problem. Items encompassed activities described in most models of metacognition in mathematical problem solving: awareness of processes before, during and after problem solving. A fourth category included a list of general problem solving strategies, e.g. I drew a picture.

Theoretical Framework

The theoretical framework for the metacognitive statements was developed from research that has examined mathematical problem solving (e.g., Garofalo & Lester, 1985 ; Schoenfeld, 1985) and from research and models in psychology on metacognition and problem solving (e.g., Flavell, 1979; Corno & Mandinach, 1983).

These studies have led to a general set of categories within higher-order processes that describe when and how an individual accesses and applies relevant information for efficient and accurate problem solving (Artz & Armour-Thomas, in preparation).

Methodology and Data Sources

In the development of the metacognitive statements several of these sources were used to generate statements that junior high school students could respond to following the working of a non-routine mathematics problem. These statements were reviewed by mathematics teachers for their meaning in instructional planning. They were also tried out with students individually, to check the meaning of the wording for students. The result was a set of 21 statements grouped into four categories: general problem solving strategies; and specific strategies used before, during, and after solving the word problem. The statements were included on one of three forms developed for a larger project, the Mathematics Assessment Project (funded by the Ford Foundation). The three forms were randomly distributed within classes at eleven junior and senior high schools in New York City during April, May and June, 1988.

The sample for the factor analysis described here included 421 students from grades 7, 8, and 9. The sample was approximately 1/3 Black, 1/3 Hispanic, and 1/3 White, with approximately 60% females and 40% males. Forms were administered during one classroom period by classroom teachers with project staff as monitors.

The data were analyzed using the SPSSX principal axis factoring with varimax rotation. Eigen values greater than 1 and a scree test were used as a criterion to determine the number of factors to retain.

Results and Conclusions

Table 1 presents the main results for the four factor solution across the complete set of 21 items. As shown in Table 1, the four factor solution provided the best interpretation. The four factors followed the general structure used in writing the items, identifying clusters of items that indicate student awareness of processing activities before, during and after solving a non-routine mathematics problem as well as general problem solving strategies.

The results of the factor analysis indicate that students' responses to being asked to reflect on their cognitive processes during problem solving have a structure that follows at least partly the general activities proposed by researchers in mathematical and psychological problem solving. In addition to the item responses obtained in this study, small-scale studies within single classrooms provided an opportunity for students to write in reactions. Of considerable interest for linking research and mathematics education are student comments such as, "It made me think about what I actually do every time I do a math example." In other pilot work it was found that the mathematics problem used with the statements is critical to eliciting awareness of cognitive processes--use of routine problems will not elicit awareness of cognition and results in different response patterns. Students appear to "go onto automatic pilot" with very familiar, routine problems.

Importance for Psychology of Mathematics Education

The importance of the results are in providing a prototype paper and pencil instrument that begins to tap student awareness of their cognitive activities during problem solving. While responses cluster into categories similar to those suggested by research, there is no intention that these be developed as a scale or scored. Future research will examine the relationship of the responses to the metacognitive statements to level of mathematics achievement. The development of an assessment instrument provides opportunities for examining the relationships of these statements and other psychological variables, and for studies of teacher use of such information in planning for and carrying out instructional activities in mathematics. The classroom use of statements such as these offers teachers an opportunity to begin to explore student cognitive activities. The statements could be used as instructional materials by directly discussing them with students, with no "scores" or "scales" implying value judgements.

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Table 1

Factor Loading for 21 Metacognitive Items
Administered to 421 Students in Grades 7, 8 and 9

	Loadings			
	I	II	III	IV
Items				
I felt confused and could not decide what to do.	**.719**	-.187	.034	-.049
I did something wrong and had to re-do my step(s).	**.538**	.026	.068	.132
I "guessed and checked."	**.521**	.066	-.068	-.022
I tried to remember if I had worked a problem like this before.	**.339**	.203	-.039	.287
I thought about a different	**.312**	.067	-.054	.198
I looked back to see if I did the correct procedures.	.006	**.686**	.227	-.028
I checked to see if my calculations were correct.	-.094	**.536**	.116	.267
I looked back at the problem to see if my answer made sense.	.127	**.496**	.077	.163
I went back and checked my work again.	.075	**.409**	.181	.235
I kept looking back at the problem after I did a step.	-.004	.055	**.578**	-.055
I drew a picture to help me understand the problem.	-.318	-.009	**.468**	-.085
I thought about all the steps as I worked the problem.	-.106	.189	**.427**	.143
I checked my work step-by step as I worked the problem.	-.116	.329	**.407**	.184
I had to stop and rethink a step I had already done.	.266	.050	**.371**	.163

	I	II	III	IV
Items				
I read the problem more than once.	.061	.111	**.290**	.078
I thought to myself, Do I understand what the question is asking me?	.033	.081	**.206**	.194
I tried to put the problem into my own words.	.144	.040	.102	**.443**
I wrote down important information.	-.076	.183	.164	**.409**
I asked myself, Is there information in the problem that I don't need?	.232	.085	-.109	**.373**
I thought about what information I needed to solve this problem.	-.019	.149	.192	**.332**
I picked out the operations I needed to do this problem.	.179	.271	-.009	**.325**

Bold type indicates highest loadings.

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TEACHER BELIEFS

BELIEFS ABOUT THE CAUSES OF SUCCESS AND FAILURE IN MATHEMATICAL PROBLEM SOLVING: TWO TEACHERS' PERSPECTIVES

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This study reports evidence of beliefs held by two high school basic skills mathematics teachers observed while solving mathematical problems. In particular, beliefs about attributions of success and failure are related to a variety of achievement and performance outcomes. The objective of this study was to document evidence of attributions of success and failure and their relationship to the problem-solving behavior exhibited by these two teachers. Observations revealed that 1) attributions were made as explanations of their performance and 2) attributions of success were classified differently with respect to the locus of control dimension of causality while attributions of failure were classified similarly for the stability and controllability dimensions of causality.

Teaching is a modeling process in which an assortment of beliefs are continually being communicated from teachers to students and are exchanged between teachers and students via a variety of behaviors. These beliefs are often communicated or interpreted as expectations and tend to shape corresponding beliefs and behaviors in students over time (Brophy & Good, 1974). Some of these beliefs are productive (constructive) and promote teacher and student behaviors which facilitate learning. Others are nonproductive (nonconstructive) and promote teacher and student behaviors which hinder learning. Thus, identifying the beliefs teachers possess which enhance or adversely affect learning is an important issue.

This paper will present findings regarding one aspect of a larger study of several achievement-related teacher beliefs (Najee-ullah, 1989). The beliefs of two high school basic skills mathematics teachers regarding the causes of their success and failure at solving mathematical problems will be reported. Such beliefs, called attributions of success and failure, have been shown to be associated with achievement-related behaviors and performance outcome (Weiner,1972, 1975, 1979; Weiner, Heckhausen, Meyer, & Cook, 1972). This research attempts to link theoretical work in psychology (attribution theory) with theoretical work in mathematics education (beliefs theory) to better understand mathematical thinking and performance. The specific research question posed was:

> What evidence of attributions of success and failure are observed in high school basic skills mathematics teachers while solving mathematical problems?

THEORETICAL BACKGROUND

Causal attributions are those factors which individuals believe to be responsible for success or failure experiences. They have been shown to guide and influence subsequent achievement-related behavior (Dweck & Goetz, 1978; Dweck & Wortman, 1982; Graham, 1986; Weiner, 1972, 1975, 1979). Individuals make attributions about their own successful or failing experiences and they also make attributions about the success and failure experiences of others (Graham, 1986; Weiner, 1972, 1975, 1979). The beliefs that teachers have about the causes of their performance or achievement outcomes are reflected in their behavior in achievement-related situations. The attribution theory of success and failure assumes that individuals actively seek reasons to explain their success and failure in achievement situations, particularly their failure (Dweck & Goetz, 1978; Dweck & Wortman, 1982; Graham, 1986; Weiner, 1975). Attributions are classified according to dimensions of causality (locus of control, stability, and controllability) which have been shown to mediate a variety of achievement-related behaviors such as goal expectancies, speed of performance, initiation of achievement tasks, value for similar tasks, persistence when faced with failure, and sympathy or anger from others.

DATA SOURCE AND METHODOLOGY

The present study used data from the Problem Solving and Thinking Project (Schultz and Hart, 1989). Data were gathered from videotaped individual and small group pre and post problem-solving sessions for two subjects, Gail and Marsha, both high school basic skills mathematics teachers who participated in the Winter 1987 Problem Solving and Thinking Institute which promoted metacognitive awareness and activity.

The problem-solving videotapes were studied in conjunction with verbatim transcripts of these sessions. The analysis of the transcripts followed the constant comparative method of analyzing data (Lincoln & Guba, 1985). This method involves the examination of data for categories of emerging patterns and themes. The Ethnograph, a computer data-management program, was used to code and catalogue the beliefs which emerged. Coded transcripts of pre and post individual and small group sessions resulted. Every excerpt identifying a belief related to the research questions was reported and examined according to type and chronologically to preserve any pre and post differences that may have occurred.

RESULTS

Attributions of success and failure will be reported for each subject separately. Session summaries will be provided followed by one example of typical excerpts from transcripts, due to space constraints. The problems worked for each session are as follows:

> Pre interview: (a) Give me two fractions whose difference is 2/13. (b) Give me two fractions with unequal denominators in lowest terms whose difference is 2/13.
>
> Pre small group: A Proper Fraction: I am a proper fraction. The product of the numerator and denominator is a multiple of seven. Their sum is a perfect square. Who am I?
>
> Post interview: There are 15 students in this class. (a) How many seating arrangements can be made with 15 desks? (b) Make sense of your answer.
>
> Post small group: In a certain card game, one of the hands dealt contains exactly 13 cards, at least one card in each suit, a different number of cards in each suit, a total of five hearts and diamonds, a total of six hearts and spades, and exactly two cards in the trump suit. Which one of the four suits is the trump suit?

Attributions for Gail

Gail made attributions for her successes and failures, but made them more often for her failures. Except for one instance, attributions were expressed in the interview sessions.

For the pre interview Gail was able to solve part (a) and made no attributions for her success. She was, however, unable to find appropriate combinations to satisfy the problem conditions in part (b), using trial and error as her primary strategy. She became frustrated and ended her problem-solving attempt. She attributed her failure to solve the problem to its being unsolvable, lack of effort, and inexperience with unsolvable problems.

> Lack of effort: For how much energy I am willing to put in on problems usually, I'd say it's, well it's still not hard because I still think that I have enough to play with, I would still want to play with it... I don't feel like, I think I've probably intelligently explored all my options. I haven't really gone crazy on it yet. There is a depth to which I will sink on these things.

For the post interview Gail thought she solved part (a) correctly, although she solved the problem using a "factorial by addition" process which she derived. She attributed her "success" to her ability to generalize and derive correct formulas by identifying patterns. Gail was unable to satisfy part (b) by explaining the factorial process. She readily admitted that her depth of understanding was superficial and that she would be unable to convey a deeper understanding to her students. She did however express success in her ability to teach students heuristic strategies such as pattern identification.

> Problem-solving ability: So being able to simplify... a problem. That's a skill. I mean... you have to acquire the ability to know that your problem will fit in the pattern of that... Knowing the factorial eliminated a lot of... long work which I would have done.

The one attribution for success made during the post small group session was ascribed to group effort or the ability of the group to work together.

Attributions for Marsha

Marsha's attributions of success or failure were made primarily during the pre interview session. Marsha made attributions for her successes and failures, but like Gail, made them more often for her failure.

In the pre interview, Marsha was able to solve part (a) and attributed her success to the easy task of working with like denominators and her appropriate strategy choice. She relied exclusively on trial and error as a strategy for part (b) and was unable to solve it, attributing this to the difficulty of working with unlike denominators, on the unfamiliarity with the interview environment and the interviewer, and her poor choice of strategy. Marsha also attributed her general lack of mathematical skill to her minimal mathematics background.

> Environment/Interviewer: If I had had longer and the camera... I don't know you... I probably could have solved it.

In the pre small group session, Marsha and her partner pursued two separate solution strategies. The frustration caused by lack of progress prompted Marsha to attribute her failure to a distraction from the camera. Marsha seemed to almost stumble upon the answer and attributed her success to chance/luck.

> Chance/luck: And then it just dawned on me that 2 and 7 is 9 and 9 is a perfect square.

During the post interview the one attribution for failure that was made by Marsha was due to an incomplete understanding of the problem.

CONCLUSIONS

In most cases when evaluating their performance, both Gail and Marsha included an explanation of their performance in the form of attributions. These were provided for evaluations of poor performance, but not necessarily for successful performance. The most common attribution was ability.

Gail primarily attributed her success and failure to internal and stable factors. Factors causing her success tended to also be uncontrollable. Internal attributions would suggest that Gail takes responsibility for her problem-solving performance and, in instances of success (especially when attributed to effort), would tend to initiate subsequent achievement tasks. The stability of her attributions for success and failure would cause her to expect similar outcomes in similar problem-solving situations, and in the case of failure, would discourage problem involvement and persistence.

Marsha's attributions of success were primarily external, while no pattern was shown for attributions of failure, suggesting that she tends not take responsibility for her successful problem-solving performance. Factors causing her success were primarily unstable. In contrast, factors causing her failure were typically stable. This would imply that she would have no strong expectations for success and have strong expectations for failure when confronted with similar problem-solving situations in the future. She would also tend to be discouraged when solving problems and lack persistence when confronted with failure. Attributions for success and failure were uncontrollable, which, in the case of failure could have been attempt to decrease adverse evaluations from others.

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PROSPECTIVE ELEMENTARY TEACHERS' BELIEFS ABOUT MATHEMATICS

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This study deals with prospective elementary teachers' beliefs about mathematics and about mathematics learning. Attitudinal surveys were used to identify a diverse sample of 22 students from 186 students enrolled in an "How Children Learn Mathematics" course. The informants were interviewed and asked to respond to a questionnaire which helped to identify beliefs about mathematics and teaching mathematics. Changes in beliefs were evidenced over the semester.

As the collection of educational research regarding teachers' knowledge grows, it becomes increasingly evident that teachers' beliefs about teaching and learning influence their practice (Thompson, 1984). Jones, Henderson, and Cooney (1986) found apparent conflicts between what teachers say they believe and what they perceive needs to occur in their classrooms. The presence of conflicts was compounded by the lack of commitment by the teachers to a coherent philosophy of mathematics teaching (Jones, Henderson & Cooney, 1986).

While novice and expert teachers may differ in their degree of effectiveness and in the depth and breadth of content knowledge and pedagogical knowledge, many prospective teachers lack the vision of what facilitating the learning of mathematical ideas entails. Germane to this vision (or lack of it) is a set of beliefs of what teaching and learning are or should be. With the wide-spread interest in teacher beliefs in general (Underhill, 1988), there is a growing body of research in terms of practicing teachers' beliefs. However, there appears to be a scarcity with regard to research reporting prospective teachers' beliefs, particularly those beliefs about learning and teaching mathematics held by prospective elementary teachers. The purpose of this study is to try to determine beliefs that prospective elementary teachers have about mathematics and about the learning of mathematics. Knowledge of these beliefs will guide teacher educators in developing for prospective elementary teachers appropriate experiences which, if necessary, will help to facilitate change in beliefs held about mathematics. While it is recognized that teacher's beliefs are influential on the manner in which a person conceptualizes his or her roles (Tobin, 1987), it is important to consider beliefs about the nature of the subject to be taught, that is, mathematics (Nespor, 1987).

PROCEDURE

All degree-seeking undergraduates at Florida State University have to complete, as part of university-wide requirements, six hours of mathematics, generally completed during their basic studies (first two years of college). Following the completion of basic studies, a student continues with requirements within a college program. Thus, most elementary education majors have completed the necessary mathematics courses prior to moving into the program. For the majority of these students, one of the courses taken is College Algebra. The second course varies, however, few of the students take a course higher than precalculus. Thus, their next encounter with a "mathematics" class is a required course taken during the second semester of their junior year. Informants for this study were enrolled in the course, "How Children Learn Mathematics." A major goal of the course is to empower prospective teachers to become learners of mathematics and keen observers of student learning rather than attempting to "teach" every topic of the elementary school curriculum.

All 186 students enrolled in the seven sections were asked to respond to a survey that was composed of 36 statements about mathematics, that is, the nature of mathematics and their feelings about mathematics. Students responded to the degree with which they agreed or disagreed with each statement. From the pool of students who were willing to talk further about their responses, 22 informants were chosen. Care was taken to select as diverse a group as possible. Interviews were conducted where informants discussed their perceptions of mathematics, influences on their attitudes about mathematics, and how they viewed the teaching and learning of mathematics. Informants were then asked to complete a questionnaire which was designed to allow for more consideration of the nature of mathematics and learning and teaching mathematics. Seventeen informants completed the questionnaire.

BELIEFS ABOUT MATHEMATICS AND KNOWING MATHEMATICS

Upon entering this course, the informants appeared to hold a formalist view about mathematics. When asked, "What does mathematics mean to you?", many responded that it was figuring out problems and coming up with solutions. One student noted, "...the way we learned it in school, which was like two plus two is four...you have a problem, you solve it, you have a right answer." Descriptions of "what mathematics is" included, but were not limited to, "formulas," "operations," "computation," "figuring out equations," "procedures," and "rules."

The belief that mathematics is procedure oriented and involves rote memorization, stated by several informants, was associated with feelings of fear and frustration. This was evident in the comment by an informant, "...what comes to my head when you mention math [is] fear. Me and numbers don't get along....I guess [I feel that way] because I never truly understood it and there's a lot of memorizing and math seems progressive so if I don't get the basics you are lost forever. Fear, anytime I have to take a math test." The fear attribute tended to affect their beliefs about what mathematics they would be able to teach successfully. The informants held a strong desire to teach only primary grades (kindergarten through third grade) because they believed the mathematics encountered in upper elementary grades would necessitate the teaching of concepts of which they had limited understanding. Foremost in their minds was the belief that the teacher had to have all the answers to the problems that would be given to elementary students and when there were areas for which the teacher did not have the answers, this topic or problem would be omitted.

BELIEFS ABOUT LEARNING MATHEMATICS

While most of the informants ascribed to the belief that mathematics was a set of rules to be memorized and followed, whereby the right answer is then obtained, it was not evident that this belief was present in their views about how children learn mathematics. Four hypothetical teachers' views were presented and informants were to identify their degree of strength of agreement with each position by allocating 100 points between the positions. There was strong agreement with positions that advocated that learning should occur with understanding and therefore involve development of the relationships one needs in order to use and understand mathematics. Ten of the seventeen agreed with the position that learning mathematics occurred during exploration--students explore problem situations, make conjectures, and discover things for themselves, thereby learning the mathematics and how it is used. Three others also agreed with this position but also believed that mathematics is learned through reasoning logically and seeing how one mathematical idea relates to another. Two informants held the belief that mathematics is only learned through reasoning logically. The last two informants indicated that learning occurred as described in the reasoning position and also supported the position that mathematics is learned through continual practice until the mathematics is "down pat."

DISCUSSION

The preliminary analysis of the data from these informants during their first mathematics education course suggests that prospective elementary teachers believe mathematics is "rule-oriented" and there is only one right answer. Presentations and discussions during the semester in "How Children Learn Mathematics" forced the informants and other students to deal with their beliefs about the nature of mathematics and the learning of mathematics. Prospective elementary teachers were challenged on what it means to be engaged in mathematical activities. Foremost was the challenge to their belief that there had to be "one way to do it, one right answer." The prospective teachers were given opportunities to view episodes where children were making sense out of the mathematics while, at the same time, they (prospective teachers) were having to make sense out of mathematics, vis-a-vis non-routine problem solving tasks. Prospective teachers began to reconsider their beliefs about mathematics. Emerging was a view of mathematics as a study of patterns and relationships which children should learn in a manner which makes sense to them (children).

While noticeable changes were evident in the fifteen weeks, further investigations into the belief systems held by these prospective teachers are needed. Informants will be followed during their second mathematics education course and, with those possible, during their internship. Data will be gathered to help describe further any changes in beliefs in mathematics, along with how the beliefs about mathematics and about learning mathematics effect their teaching of mathematics.

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PROSPECTIVE ELEMENTARY TEACHERS' CONCEPTIONS ABOUT THE TEACHING AND LEARNING OF MATHEMATICS IN THE CONTEXT OF WORKING WITH RATIOS

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ABSTRACT

Though the preservice elementary teachers in this study generally knew how to solve the mathematical problems given to them, they had little to say about the validity of different methods. To understand their pupils' methods of solution should be important to them, but this may not be so, given their views about teaching mathematics.

This study is part of a research project which set out to address issues such as: kinds of explanations in mathematics given by preservice elementary teachers, the mathematical knowledge these explanations rest on, the subjects' beliefs about mathematics and the teaching and learning of mathematics that can be inferred from their explanations. Research relevant to the project includes studies on understanding in mathematics (Davis, 1984; Hiebert (Ed.), 1986), and on beliefs about mathematics (Cobb, 1985; Schoenfeld, 1985). The present study was carried out in the framework of an eight week summer course in mathematics for elementary education majors. Instruction was conducted in small groups in which the students were encouraged to participate actively in the construction of mathematics through peer dialogue (Bishop, 1985; Lampert, 1988).

Eight students-subjects were enrolled in the course, six seniors and two graduate students--all female. Sources of data included: tape recordings of the students' in-class group work and of task-based interviews; written homework problems, essays, their diaries, and in-class observations.

This paper focuses on two situations involving proportional reasoning as the context for presenting some of the subjects' views on mathematics and on teaching mathematics. At the time of the two situations presented below, proportional reasoning had not been discussed as part of the course.

DISCUSSION

Dionne (1985) discusses the weight that teachers give to product and to process in mathematics. Inspired by that issue, I assigned as homework the task below, adapted from Hirabayashi (1985); in this scenario a fifth grader reaches the correct answer via a faulty reasoning.

A group of fifth graders were working on the following problem:

Three children are practicing basket-ball shooting; this is the table recording the results:

player	shots	successes
A	4	2
B	10	4
C	20	6

Question: who is the best player?

One of the fifth graders in his solution to the problem made the following table:

player	shots	successes
A	20	18
B	20	14
C	20	6

He then concludes that A is the best player.

You are asked to comment on this student's work.

Five subjects disagreed with the child's work, two agreed, and the last subject could not figure out what the child had done. All the subjects who disagreed wrote how he should have done the problem. The strategies used were percentages, unit rates, and comparison of fractions. Their responses do not convey any shade of hesitation on their part: they all seemed very confident about the way to approach this problem. But, I cannot help wondering how solid their confidence was: when further probed, only one out of the five subjects was able to produce a convincing argument for her work.

As to the two subjects who agreed with the child's work, it is interesting to note that both of them offered similar justifications for the child's procedure - namely that the ratio or the relationship did not change:

> The way this child resolved this problem was to find a "common ground" for all the shooters. ... To do this, he added 10 to B's amount of shots and 10 to his successes. This is appropriate and good because if he did not, the original 10 to 4 ratio would not be the same. ... On the whole, I think that this approach to the problem was creative, divergent, and insightful. [Carol]

Donna wrote:

> The fifth grader had a brilliant idea. He decided to use a common number of shots and adjust the successes accordingly. ... he took the difference of 20 - 2 for player A, and 20 - 6 for player B to

figure out their success; the relationship is the same so it doesn't change the problem. [Donna]

Did Carol really think that 10 to 4 is the same ratio as 20 to 14? What did Donna mean by "the relationship is the same"? It is interesting to note that both, Carol and Donna, knew how to solve this problem with correct proportional reasoning. Carol had done so in her written homework, before commenting on the child's work. Donna did so in an interview before getting the homework back. Why then did they agree with the child's work? Was it the fact that he had the right answer? Did they think that this was an alternative procedure?

Answer versus procedure

The fact that the child got the right answer is likely to have led Carol and Donna to agree with the child's work. In fact, before handing back their homework I interviewed Joyce (the subject who had not been able to figure out what the child had done) and Carol together:

Carol: I think it's totally appropriate, like any way he gets to it is fine; I personally went in terms of ratios and percentages, because that's the way I think; but it's just an alternative means to the solution, so I would think that it would be good.
Joyce: Because his answer was right?
Carol: Yeah, but also because
Joyce: I mean, would you point out his error or not?
Carol: Yes [very definite] I would; I would say this is a really excellent way for you to approach this and I'm really proud of you for thinking on your own, and I really value that you did that; however you did stumble a little bit ...

However, it turns out that what she meant by stumbling was not the procedure itself, but the fact that she had misread one of the child's numbers, and thus thought that he had made an arithmetic mistake (for player A she had read 20 to 16 instead of 20 to 18).

I also interviewed Donna on this problem, and gave her the following table:

player	shots	successes
A	5	1
B	10	4
C	20	10

Using the child's procedure she concluded that A would be the best player, but remarked that using her method C was the best player. She commented:

Donna: I don't really know. Is he doing something wrong?...Well, I

> thought it was good, obviously, and I said it was; well for this particular case it worked out; maybe it won't work out for everything.... I don't know how to explain how come it's wrong.

Donna's observation that the child's procedure "worked" in this case was also shared by one of the five subjects who pointed out that the child had done something incorrect: "His simple logic works in this case." [Vicky]

What I have presented above leaves me wondering how ready these prospective teachers are to understand children's work. How are they going to handle it when one of their students comes up with a method different from theirs? What means do they have to determine the validity of a method?

Teaching mathematics to children

Even for the subjects who did not agree with the child's procedure, most of them made sure to point out something that they thought was positive in the child's work, for example that he had thought of getting a common denominator, or that he had got the right answer. Three prevalent threads in these prospective teachers' thinking about teaching mathematics to children are:

- To avoid the child's frustration:

> Joyce: If he was really stumbling and got to the point of frustration, I would point out that it's half, because you don't want to get them too frustrated because then if just turns them off to math and you don't want to do that either.

- To praise his work:

> I would comment that the student was right to change all the shots to 20 since that was a common number (emphasize good). [Ann]

- To encourage him for thinking on his own:

> Carol: I'm really proud of you for thinking on your own.

The subjects' view of their role as teachers is to show the child what to do, to avoid confusion and to straighten him out, while giving him credit for trying and doing something on his own. As Ball (1988) points out, "thinking on your own" may not always be something to praise if in conflict with sound mathematical thinking.

Handling conflict

The basketball task left me wondering about what kinds of arguments the subjects could offer against inappropriate additive strategies in ratio situations.

A couple of weeks later, I audiotaped a session in which six subjects worked in a group setting on the following problem:

> If you need 1 and 1/3 cups of sugar and 4 cups of flour to bake a cake, how many cups of sugar will you need if you want to use 7 cups of flour?

Four of the subjects solved the problem via a well rehearsed procedure: set the two ratios to be equal and solve for x. Another subject, Vicky, used a more informal, everyday mathematics type of reasoning (Lave, Murtaugh & de la Rocha, 1984). She drew out the cups of flour and sugar, and immediately saw the relationship between the amount of sugar and that of flour (i.e. 1:3). The last subject, Carol, solved the problem using an additive strategy [7-4 = 3, so we need to add 3 cups of sugar, getting 4 1/3 cups of sugar for 7 cups of flour]. The other members in the group tried to explain why that strategy was not appropriate. Their efforts left me with an feeling of uneasiness at the thought that these subjects were about to become teachers. The subjects could barely go beyond saying "but you can't add 3". Every further comment that they offered was either a paraphrase of that one, a misunderstanding of what Carol was doing, or one based on their own correct procedure.

In a last attempt to justify why addition was not appropriate, Lisa told Carol to change the 1 1/3 to 2, to make it easier and they worked out the problem using addition. This gave them 2 to 4 and 5 to 7. The discussion ended as follows:

Betsy: Two is half of 4, right? Is 5 half of 7?
Carol: No.
Betsy: Right, so you know it's wrong.
Lisa: Do you understand that?
Carol: Yeah, I understand that. [in a rather submissive tone]

However, I do not think that Carol understood. For her it was once more an authority telling her that what she had done was wrong. She usually tried very hard to understand and was not willing to "just accept"; but by that point in this problem, she was ready to give up. Earlier in that same problem, Lisa had shown Carol how she had set up her problem as an equality of two fractions while Carol had set it up as 1 1/3 : 4 and _ : 7. The following dialogue ensued:

Carol: I had no idea that a fraction was like a ratio.

Lisa: You did have an idea, though; you just got confused; you had the right idea; these two dots mean ratio technically.
Carol: But I didn't know that a fraction was the same thing as a ratio.
Lisa: Well, you can set it up this way [with the two dots] if you want.
Carol: But weren't you writing this as a fraction?
Betsy: To solve it.
Lisa: Yeah, to solve it, just to solve it.
Carol: I didn't even know that 3 : 4 is 3/4.

I do not think that either Lisa or Betsy realized the extent of Carol's confusion. To both of them it was "obvious" that fractions and ratios were interchangeable. It was not obvious to Carol as she insists on that point in her four statements in the excerpt above. Carol's use of an additive strategy and her not knowing that "a fraction was like a ratio" deserve closer attention than her peers were able to give, as the research on proportional reasoning illustrates (Karplus, Pulos & Stage, 1983; Koch, 1987).

CONCLUSION

The fact that all the subjects knew how to solve the basketball problem is not very encouraging once we look closely at their understanding. Accepting a method of solution because it happens to yield the right answer, or rejecting it because it differs from their own, are issues which need to be addressed in teacher education. Most of the subjects could not explain why the method they had used worked, nor why another method was not appropriate. That their understanding of ratios was rather superficial was, unfortunately, not surprising given the subjects' mathematics background characterized by an emphasis on memorization of rules with little conceptual understanding. What I found more troublesome is that the subjects did not seem to think that as teachers they needed to know about these issues.

To them teaching was essentially showing the child how to do it, while avoiding his confusion or frustration. These subjects were not prepared to look at the learner's work from his perspective, nor were they aware of the issues involved in the learner's sources of conflict. To address these, they resorted to an authoritative "this is wrong, this is how it should be" type of talk; or, they tried to comfort the learner by "putting the blame" on the mathematics, as illustrated in the following comment about the cups of sugar/flour problem:

Lisa: Ok, it's just a little bit more confusing when you have fractions because it's hard to figure it out.

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CHANGING PRESERVICE TEACHERS' BELIEFS ABOUT MATHEMATICS EDUCATION

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This study examines an intervention in an elementary teacher education program. (1) The intervention - a sequence of mathematics courses and an integrated methods course - emphasizes the conceptual foundations of mathematics. The paper investigates some of the changes in teacher candidates' beliefs about how mathematics is learned, what it means to know mathematics, and the role of the teacher in creating effective mathematical experiences for children.

PRELUDE

> Learners of mathematics taught in an environment which emphasizes only the acquisition of facts and conventional algorithms have but one recourse when confronted with a mathematical problem: searching their memories for the fact or the procedure. School mathematics becomes the accumulation of huge numbers of problems, algorithms for their solution, mystical formulas and misremembered bits and pieces. Contrast the freedom and power of a mathematics learner nurtured in an environment in which the goal is to learn ways of finding out, ways of making sense of mathematics and strategies for inventing procedures to solve problems or for building models to understand mathematical situations.

Reflecting on the prelude leads one to the questions, "What teacher knowledge is needed?" and "What contextual conditions foster the creation of such places for learning mathematics?" With these questions in mind, we designed an improvement study - a sequence of courses on number, geometry, probability and statistics, and a coordinated methods course - which had as a basic goal demonstrating the feasibility of creating in new teachers a more conceptual level of knowledge about mathematics, mathematics learning and mathematics teaching. The longitudinal study is guided by the overarching question, "What is the nature and the extent of changes in the beliefs and knowledge about mathematics and mathematics education among preservice teachers as a result of the intervention?"

An earlier PME paper discussed findings from the first term of the intervention (Schram & Wilcox, 1988). The report presented here represents an additional year of data collection and a more developed theoretical framework for analysis of the data. Students have completed the first two math courses, the methods course and student teaching and are currently taking the final course in the mathematics sequence.

DATA SOURCE AND ANALYTICAL FRAMEWORK

We have collected classroom observation and questionnaire data from a cohort of 24 preservice elementary teachers who entered the Academic Learning Program at Michigan State University in September, 1987. We have additional data from an intensive sample of four students that include tape-recorded interviews, writing assignments, observations of their student teaching and tape-recorded conferences with mentor teachers and fieldwork instructors. We have developed an analytic framework for the three questions that inform this paper: 1) What does it mean to know mathematics? 2) How is mathematics learned? 3) What is the teacher's role in creating effective mathematical experiences for children? The framework describes three levels corresponding to different orientations to teaching and learning mathematics. Our earlier paper provided the expanded framework for the first two questions. An abbreviated framework for the teacher's role is presented in Figure 1. The levels provide a way to analyze changes in students' beliefs about teaching and learning mathematics as they progress through their teacher education program.

What is the Teacher's Role in the Mathematics Classroom

Level 3:	Establishing mathematical goals that emphasize conceptual development and relationships; providing problem situations that lead to learner explorations and inventions; creating opportunities for children to talk with each other about mathematics.
Level 2:	Establishing mathematical goals that emphasize understanding procedures; providing activities that are interesting but routine; asking questions that require an explanation of procedures.
Level 1:	Carrying out goals as determined by text material; providing demonstrations and examples of tasks to be completed; checking assignments for completeness and accuracy.

Figure 1: Levels of Orientation

DISCUSSION

Over the two years in which we have been conducting our study, we have seen significant changes in the teacher candidates' beliefs about what it means to know mathematics and how mathematics is learned. But in the context of student teaching, we uncovered a tension between their ideal vision related to themselves as adult learners of mathematics and their practice with young children. The first section of this paper illustrates how members of the cohort were coming to think of themselves as learners of mathematics. The second section offers a contrast in how two of our subjects interpreted the teacher's role in the mathematics classroom during their student teaching experience.

Teacher candidates as learners of mathematics. When the teacher candidates entered the first course of the mathematics sequence, memory and algorithmic thinking played a large role in their conception of what it means to know mathematics. By the second mathematics course, they were developing a set of intellectual tools - ways of thinking about problems, a repertoire of strategies, models and representations, and a disposition to engage collectively in mathematical searches - that increased their confidence and ability to apply knowledge in unfamiliar problem contexts. There was the beginning of a shift away from the instructor as the sole authority for knowing and a reliance on their collective ability to decide when a problem had reached resolution. The following instance exemplified this attitude.

In the second week of the geometry course, the class was presented with this problem:

> There is a tree at every lattice point on a geoboard that is extended infinitely to the right and up. You are standing at (0,0). Is it possible to begin at (0,0) and walk on a straight path through the forest without hitting a tree?

Students' intuitive sense was that one would eventually walk into a tree. The initial discussion centered on various ways to think about the problem. Several students thought it might be approached by considering the slope of a line. Others argued for a consideration of the notion of infinity while others offered suggestions about angle measures. The charge to the students was to think hard about the various conjectures before the next class. On the second day, much of the discussion centered on struggling with the idea of infinity.

> Lori: You couldn't walk without hitting a tree because it is impossible to go at an infinitely small angle. It would be infinitely small, but once you draw it, it is finite. You could never start the walk.
>
> Instructor: Is it fair to say that what you wanted to do was something very close to the x-axis? Now you are saying that won't work because eventually this thing moves up into the forest?
>
> Bonnie: Visible points are ones that are relatively prime coordinates. So if you go to just primes, there are an infinite number of primes so eventually you have to hit a tree.
>
> Instructor: You're saying...if this set is infinite, there must not be any holes in it. Can you think of any sets that are infinite that have holes in them?

These conjectures led to a discussion of rational numbers, a return to the notion of slope, and a consideration of lines with rational and irrational slopes. Not yet convinced by any argument, the class put the problem aside, returning to it the next meeting.

> Lori: If you went to the point (1, 2) like we were talking about, if you draw a line...(pause)...like that's never going to get you to a point because 2 times any whole number is never going to give you an integer.
>
> Instructor: (drawing the line Lori was talking about) Is it possible for that line to hit a lattice point where the coordinates are whole numbers? It is sometimes easier to understand an argument if you look at the negative side of it. If that line hits a lattice point, what can you say about the slope of that line?

The students reasoned that if the line did hit a lattice point, the slope would have a rational representation which was a contradiction.

These students wrestled with an unfamiliar problem over several days. Some ideas were advanced, abandoned for a while, then returned to as new insights were revealed. The inquiry continued until students convinced themselves with a persuasive and mathematically reasonable argument that it was indeed possible to walk not just one path, but an infinite number of paths, without hitting a tree. We saw this pattern throughout the geometry and probability courses--their increasing conceptual orientation to the study of mathematics.

The teacher's role in the mathematics classroom. Many of the teacher candidates appeared to be moving toward Level 3 in the way they talked about the teacher's role in creating effective mathematical experiences for children. Because of their experience in the math sequence, they talked about group work, non-routine problem situations and multiple representations as a powerful way for students to explore mathematics and construct mathematical knowledge. However, in the context of student teaching, our observations revealed a tension between this ideal vision and their practice with children. We draw on data for two of our subjects to compare their approaches to planning and carrying out instruction.

Linda. Linda wrestled with the big conceptual ideas embedded in a particular piece of mathematical content.

> Linda: I knew it would be important to help students construct a solid concept of fraction - what is a fraction...what they represent...a variety of situations where understanding part of a whole and part of a set are important.

She thought hard about what she wanted her students to understand.

She regularly drew from three main resources--the textbook, NCTM Standards (Commission on Standards for School Mathematics, 1989) and her own evolving conceptual framework of mathematics and how it is learned. She was critical of her textbook in which a single page was intended to develop a complex idea.

> Linda: There was one page in the text...in which a student could easily do the entire page, get 100% - all they had to do is count and fill in numbers - and not get any idea about what it really means to think about fractions as part of a whole.

She valued providing opportunities where students could talk about and make sense of mathematical ideas for themselves. We observed her use the daily "lunch count" as a problem situation. She noticed some students wrote 25/29, others wrote 29/25, to represent the fraction of students who were present that day. She raised this with the class.

> Linda: Here's a couple of different things I see people writing down. Could someone explain what this [25/29] means?...Could someone explain what this [29/25] means?...Which one of these describes the situation we have?

When the question was posed about the fraction of people getting hot lunch, controversy arose. Students debated whether the "whole" was the number of students in the class or the number present that day. They reasoned that the "whole" should be the number present because those who were absent would not be having lunch at school.

Her attempts to organize instruction around central ideas and relationships were sometimes frustrated by gaps in her mathematical knowledge. Her field instructor described it this way.

> An analogy to a road map helps me think about what is missing for Linda. She knows that a big picture exists. The big ideas can be represented by cities. But some of the roads connecting the cities seem incomplete. She doesn't always understand the subtleties. For example, in her fraction unit, she got into trouble when she introduced her representation of equivalent fractions. She didn't understand the big conceptual leap it required for kids.

Denise. Denise consistently focused on getting to "the algorithm." She was committed to beginning instruction at the concrete level. But the choices she made were driven by "neat" activities rather than an overall conception of the mathematical content or how the activity would help children to understand an idea. In some cases, the manipulative she chose was an inappropriate model for the mathematical idea. Her unit on division is illustrative. She began by having youngsters put several hundred pieces of macaroni into 2, 3, and 4 groups. On the second day, she moved to the symbolic level, emphasizing place value and partial quotients, using this form as a way to "record the answer."

For class work, students were to divide 115 pieces of macaroni into 2, 3, 4, 5, 6, and 7 groups and record their answer as above. Most children worked with the macaroni. A few resisted, preferring to just do the problems symbolically. However, these children very quickly ran into trouble. They could not figure out what to do in 3 115 when 3 would not divide exactly into 100. On the third day, Denise used a "chip trading" board to illustrate regrouping in division. Again, work at the concrete level was coupled with the symbolic "record." On the fourth day, her final day of student teaching, she demonstrated the long division algorithm at the board. In the span of four days, students had been given two different models for thinking about division as well as an algorithm. In the rush to the algorithm, there was little attempt to make connections among the various representations.

In an interview following observation of these lessons, Denise explained her decision-making.

> All I really wanted them to see out of that was an experience they could think back to when they get into the symbolic representation. I thought there were a few of them that would get the idea that if the 100 didn't divide evenly that they were going to have to do something with the extras. Now today, trading with the chips, I hope the connection is made. I hope that once it's followed through with the actual algorithmic step that this will all make more sense.

For her, getting to the algorithm was key because that would clear up any misunderstandings children had with concrete representations. Her mentor commented on this.

> She's getting so much into the algorithms. She thinks this will make it clearer for the children...To her, those symbols convey all the thoughts that she needs.

Summary. Linda thought about her role as a mathematics teacher in a way that consistently approached Level 3. Denise, on the other hand, approached Level 2. What distinguished these two student teachers were the mathematical goals they set for their students, the learning opportunities they provided, and the degree to which conceptual understanding or algorithmic thinking focused their efforts.

Efforts to improve the teaching and learning of mathematics by reorganizing the curriculum around concept development and problem solving requires that teachers have a conceptual understanding of mathematics. It also means challenging their deeply held beliefs about young children as learners of mathematics and about the elementary mathematics curriculum. Our study reveals the complexity of this change process. Over the next year, we will be following our intensive sample as they enter their first year of teaching. Will Linda be able to withstand the pressures to conform to a more traditional approach? Will she pursue ways to increase her own mathematical knowledge? What might push Denise beyond her algorithmic approach to teaching and learning mathematics? This continuing study has implications for teacher educators and mathematics educators working to reform elementary teacher education programs.

Endnotes

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TEACHER EDUCATION AND TEACHER DEVELOPMENT

LEARNING TO TEACH MATHEMATICS:
A REPORT ON THE METHODOLOGY OF AN ECLECTIC INVESTIGATION

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This paper discusses the methodology used in the ongoing investigation "Learning to Teach Mathematics: The Evolution of Novice Teachers' Instructional Decisions and Actions." This longitudinal study is concerned with the claims, thinking, and actions of 8 preservice middle school mathematics teachers, and with the sociocultural environment in which they are learning to teach. The paper reports methodology intended to provide background data with which to develop grounded theory about the learning-to-teach process.

The project "Learning to Teach Mathematics: The Evolution of Novice Teachers' Instructional Decisions and Actions" (NSF # MDR 8652476) involves a longitudinal study of beginning middle school mathematics teachers. It is our intent to gain as complete an understanding as possible of the process of learning to teach as experienced by beginning teachers and the forces that influence their professional development while they are learning to teach. Our goals are (1) to provide the kind of background data on beginning mathematics teachers that will foster the development of richer theoretical frameworks for research in mathematics teacher education, and (2) to make recommendations to the field concerning mathematics teacher education programs. We are using multiple research perspectives and methodological techniques.

In order to meet our research goals, we are investigating the teachers' *claims*--their knowledge and beliefs--about mathematics, mathematics pedagogy, general pedagogy, curriculum, learners, and themselves as teachers. We also are investigating the teachers' *thinking* before, during, and after teaching and the relationships between their thinking and their claims. We are concerned with the *sociocultural environment* our informants encounter, both in their preservice teacher education program and in the schools in which they teach, and the effects on them of participating in the research study.

During the first year of the study (AY 1988-89), we followed 8 preservice teachers, who selected mathematics as a concentration, through their final year of teacher education. This experience included several methods-of-teaching classes, three half-day student teaching placements, and one full-day student teaching placement. Our informants represent diverse educational backgrounds and a range of competencies in mathematics. We attempted to select informants, from a cohort of 36 preservice teachers, consistent with the ethnic and gender makeup of the group. The informants include seven white females and one black female. One white male initially agreed to participate but later withdrew. During the second year (AY 1989-90), we will be following 4 of the 8 informants through their first year of classroom teaching.

Our data collection procedures are numerous and draw from both the anthropological and psychological traditions. We are committed to qualitative methodology (Erickson, 1986; Goetz and LeCompte, 1984) in order to try to develop the depth of understanding we seek about individuals and about the process of learning to teach. We rely primarily upon interviews and observations in order to gather information about informants' claims, thoughts, and actions. These data are supplemented by questionnaires and written documents (e.g., lesson plans, assignments in the mathematics methods course, journals). We also are gathering interview and observational data from significant others in the sociocultural environment in which the informants are learning to teach. All interviews are semi-structured and are based on protocols developed and piloted in advance. All are audiotaped and transcribed for analyses.

CLAIMS

For this project, we are investigating the *claims* held by individuals about mathematics, mathematics pedagogy, general pedagogy, curriculum, learners, and themselves as teachers. Claims are taken to be elements on a beliefs-knowledge continuum. Both beliefs and knowledge are concerned with what a person holds to be true (Green, 1971) and are distinguished in part by the extent to which an individual can promise to others the right to be sure of the knowledge or belief claim (Scheffler, 1965). With both are associated a way of holding them, evidence for holding them, and cognitive and affective content. For this project, we are investigating the types of claims individuals hold, the bases for holding them, the susceptibility of the claims to change, how consciously they are held, and their emotive content. We are interested in how the claims interact with the teachers' thinking and actions and how the sociocultural environment in which our informants are learning to teach affects their claims.

Although we are using many data sources to try to understand the participants' claims, we designed specific interviews and questionnaires to focus most directly on this component of the research. Recognizing that claims are always embedded in a paradigm or world view (Kuhn, 1970) and that they are not always accessible for articulation, we designed these instruments to not be connected overtly to the teacher education program and to combine a mixture of direct and oblique questions. For example, in trying to learn about the claims the preservice teachers hold about teaching the division of common fractions, we asked them to respond to materials from the teacher's edition of a sixth-grade textbook, asked them to respond to a hypothetical student's homework on problems taken from that material, and asked them to describe how they would go about teaching it. In another part of the interviews, we asked them to describe the relationships they see between 19 common topics in middle school mathematics, one of which was the division of common fractions. Both the interviews and the questionnaires ask the preservice teachers to respond to what it means to **know** the division of common fractions and ask them to work problems and illustrate them with diagrams or stories. We also are interviewing and observing the preservice teachers during their field placements in order to see whether and how their claims are evident in their planning and teaching actions. Throughout our data collection we strive to "make the familiar strange" (Spradley, 1979). That is, we ask our informants to explain the source of their ideas, how and why they decided on teaching actions, and what they mean by such terms as "problem," "understanding," or "the basic idea" in order to guard against over-inferring on the basis of situations, words, or phrases that might be assumed to have common and shared meanings. These instruments and observations are used at several different points in the year, and are intended to provide us "snapshots" of the participants' claims. Thus they provide us with some information about changes in claims.

THINKING

Thinking is a way in which the teachers call upon their claims. Temporally, this encompasses three categories (Clark and Peterson, 1986): pre-active, interactive, and post-active (with respect to teaching). Teacher thinking can also be categorized according to planning, reflecting, evaluating, and responding to uncertainties or dilemmas (Clark, 1988). We are investigating the nature of teachers' thinking as categorized temporally, the resources upon which they draw in their thinking, the relationships between thinking and claims, and whether and how teachers' thinking changes over time and with experience.

Since we do not have direct access to our informants' thinking, we must rely upon self-reports and observations. We are using a number of data sources to try to understand informants' thinking and its relation to their claims. With respect to their teaching, we are observing each informant's teaching on five consecutive days at three points during the year and are interviewing them before class about their plans and how they arrived at those plans, and after class about the teaching that we observed. The pre-observation interview elicits information concerning their intended actions, how they planned for the class session (their "pedagogical reasoning," Shulman, 1987), upon what resources they drew when planning, and what they would like us to pay particular attention to during the class session. The post-observation interview focusses on the informant's reaction to the class (what she was pleased or concerned about), what she would do differently if she could reteach the class, and episodes that we feel constitute "critical moments" (Shroyer, 1978) in the class session. Critical moments include such things as explanations or examples offered to the class or to individuals and actions that appear to be taken as a result of an immediate situation. In order to answer questions concerning the influences of the informants' supervisors, interactions between informants and these significant others are observed and expectations and reflections of the informants and significant others are elicited in interviews.

We also are learning about our informants' thinking by asking them to "think aloud" during different tasks in the claims interviews. For example, they are asked to think aloud when sorting mathematical topics in order to show relationships between them; they are asked to think aloud when working mathematics problems; they are asked to think aloud when sorting phrases about types of teachers in order to tell us about characteristics of mathematics teachers, English teachers, elementary school teachers, middle school teachers, and themselves as teachers.

SOCIOCULTURAL ENVIRONMENT

In gathering data about the sociocultural environments in which our informants are learning to teach, we are concerned with two broad categories: the environment of the university experience and the environment of the school experience. The instruments we are using to gather data regarding the informants' university experiences are designed to provide a picture of how the sociocultural environment of the university and informants' experiences within this environment influence their claims and the process of learning to teach. The experiences of our informants related to their particular teacher education program are hypothesized to be major factors in their learning-to-teach process. In order to document these experiences we developed a number of interviews to be conducted with informants and with significant others. These interviews are designed to provide data concerning the various components of the program from a number of perspectives and to help us understand the informants' experiences in the program.

Data regarding the informants' school experiences (their field placements) are designed to help us understand their claims, thinking, and actions. Interviews were developed to gather information concerning the sociocultural environment of the school division, the schools, and the classrooms in which our informants taught. These interviews are conducted with informants, cooperating teachers, university supervisors, building principals, and several administrators from the school division central offices. They elicit information concerning the social organization of mathematics instruction, cultures of teaching, and significant others' claims about working with student teachers.

SUMMARY

There are several critical aspects of this investigation. First and foremost is its attention to both the cognitive and affective aspects of preservice teachers' claims and thinking **and** to the sociocultural environment in which they are learning to teach. This has provided a difficult challenge to meet in designing and carrying out our data collection. Yet we find the balance between these aspects of the learning-to-teach process to be a very natural and vital one for investigation. Framing the investigation around the domains of mathematics, mathematics pedagogy, general pedagogy, curriculum, learners, and self-as-teacher is enabling us to learn about the importance of **mathematics** in the process of learning to teach mathematics in the middle grades. It has become apparent that a less holistic approach to learning about the informants' experiences would yield a meager picture at best.

A second critical aspect of the investigation is its longitudinal design. A long-term involvement is vital to the development of any robust understanding of the learning-to-teach process. This process is *not* restricted to the university classroom or to a relatively short field placement; rather, it continues well into a teacher's full-time classroom experience. Following the teachers over a two-year period will allow us to see whether and how their claims, thinking, and teaching change and what factors are important in any changes.

Finally, it is critical that we are not involved in any evaluation of the informants, their placement schools, or their teacher education program. We are trying to understand what aspects of the learning-to-teach process are salient for them; we are not attempting to see "how well" they are doing. The project is involved with building grounded theory (Strauss, 1987) with which to inform further research and development in mathematics teacher education.

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ELEMENTARY TEACHERS AND PROBLEM SOLVING: TEACHER REACTIONS AND STUDENT RESULTS

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This paper describes a course in mathematical problem solving strategies for elementary school teachers the results of this training of their students' problem solving performance as measured by select items from NAEP, and the teachers' reactions to the course.

The results of the Fourth National Assessment of Educational Progress call attention to two areas of weakness in the mathematical performance of middle school students: they perform poorly on problems requiring non-routine and problem solving type solutions; and boys outperform girls on higher level application of mathematical skills, on geometry-related problems, and on problems in measurement (Dossey, Mullis, Lindquist & Chambers, 1988). Other research in teacher knowledge of mathematics makes a strong case that efforts to improve children's mathematics learning might first begin with enhancing teachers' knowledge about mathematics (Ball, 1988; Oprea and Stonewater, 1988). Yet the degree to which inservice training of teachers results in improved learning in their students is still an open question (Szetela and Super, 1987).

Partly in response to these research findings, The Ohio Problem Solving Consortium has received funding to form a cooperative venture between public school teachers and university personnel to implement a multi- phase project to improve the problem solving abilities of middle school teachers and students (Stonewater and Kullman, 1985; Stonewater and Oprea, 1988). The purpose of this article is to describe the problem solving course and report the results of the course in terms of the teachers' students problem solving abilities and in terms of teacher-reaction to the course.

METHODOLOGY

To assess the effectiveness of the course on the participating teachers' students, 19 sixth through eighth grade Consortium teachers and six middle school teachers not involved in the project administered select items from the NAEP before the problem solving course began, and again six months later after the end of the course. Eight items were chosen from the fourth mathematics assessment and were selected to represent problems which could be solved using at least one

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of the strategies learned by the teachers. A total of 365 experimental students and 91 control students in grades six through eight completed all testing. The sample included 209 males and 247 females.

Data were analyzed using a multiple analysis of covariance, with pre-test scores as the covariate, posttest scores on each item as the eight dependent variables, and control or experimental group as the independent variable. For statistically significant F-values, univariate post-hoc analyses of variance were computed for each of the eight test items separately. Similar analyses were done in order to analyze possible gender differences. Finally, teacher response to the problem solving course was assessed via written questionnaires and journals they kept during the class.

PROBLEM SOLVING COURSE

Teachers were expected to learn and be able to use seven different problem solving strategies: Guess and Check; Patterns; Simpler Problem; Elimination; Working Backwards; and Simulation. Teachers were also expected to reorient their own teaching to include units on each of the problem solving strategies. The problem solving course was designed on the basis of the Instructional Model for Problem Solving or IMPS model (Stonewater, Stonewater, & Perry, 1988), which is grounded in cognitive developmental theory and describes three categories of instructional approaches: structure; direct experience; and diversity. Examples from two of these categories are described below.

Direct Experience The IMPS model also suggests that activities which engage teachers in direct application of what they are learning will enhance learning. First, in conjunction with the local public broadcasting television station, a series of four video tapes, entitled *Problem Solving in the Middle School*, were developed as examples of what "master teachers" do when teaching problem solving. These were viewed by the class. One particular useful portion of the tapes shows middle school teachers actually using various problem solving strategies in their classes. Teachers particularly liked segments of the tapes which showed students working on the strategies and teachers could often relate their own students' reactions and problems to what they saw on the tape.

As another direct experience method, teachers were asked to apply each of the strategies in their own classes and to keep a journal of their experiences. While this activity did not engage the teachers directly in actual problem solving, it helped them build confidence in their abilities to teach problem solving.

Diversity Another approach used in the IMPS model is that students must actually engage in and confront the complexities of what is to be learned in order for them to experience disequilibrium. Presenting diverse situations for the teachers to engage in is a method of doing this. Thus, problems which required the teachers to generalize beyond their current levels of mathematical knowledge and thinking ability were presented. For example, teachers rarely had difficulty with pattern problems like predicting the next term in the sequence $1/(1*2)$, $1/(2*3)$, $1/(3*4)$,.... But in order to challenge the teachers and create the required disequilibrium, the problem was extended to find the sum of the series $1/n(n+1)$.

RESULTS

Effect of Participation Results of the statistical analysis of student performance on the NAEP items indicated an overall by-item difference between groups ($F=2.58$, $p<.01$). Post-hoc analyses indicate these differences appear for two items: one measuring a combination Guess and Check/Elimination strategy ($F=12.79$, $p<.01$) and the other a Guess and Check problem ($F=5.70$, $p<.02$). Both of these differences were in favor of the experimental group.

Gender Analysis There were no statistically significant differences in problem solving between boys and girls in the experimental group by item ($F=1.18$, $p<.31$). On the total test, percent increase from pre- test to post-test for girls did exceed that of the boys: 10.3% vs. 9.2%, respectively, but adjusted post-test means are almost identical for the two groups: boys answered an average of 5.34 (66.7%) out of 8 items correctly on the post-test and girls answered an average of 5.30 (66.3%) items correctly.

Teacher Reactions Teacher reactions from the post- class and post-course questionnaires, as well as from their journals, point out two major areas in which they felt they had changed: self-confidence; and self- perception as mathematics teachers:

> When I begin a Friday math class with something other than problem solving and my students remind me that Fridays are reserved for problem solving, I know I've done the right thing. When students ask for additional copies of the problems because their parents also enjoy doing them, I know I've done the right thing. When I, a person respected as a language arts teacher, lose track of math time, dismiss my math class late and only then, reluctantly, I'm SURE I've done the right thing!

Teachers felt that their increased self-confidence was primarily due to the fact that the strategies they learned provided a variety of new-found tools, methods and approaches that they could use to attack problems. Teachers also reported shifts in the perceptions they held of themselves as mathematics teachers. A number of teachers pointed out that prior to the course, they had not really thought of themselves as mathematics teachers, but identified themselves more closely with other content areas, often language arts:

> What an exhausting week-end. My mind is still spinning from our meeting. You don't know what a change this is for me. I have always been Linda, teacher of reading and other language arts. I have really worked and worked to learn all I can in this area. Now you tell me I must be Linda, teacher of math too? What really blows me away is that I am really enjoying it! I feel like the kid...who gets the award for the most improved!

CONCLUSION

The results of the NAEP study indicate that the experimental group statistically out-perform the control group on only two problems out of eight, and both problems were classified as solvable by Guess and Check. Further, there were no differences in performance between girls and boys. Finally, teachers reported positive changes in their self-confidence in doing mathematical problem solving, and recognized changes in their perceptions of their roles as teachers.

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THE USE OF 'MINI-INTERVIEWS' BY 'ORTHOPÉDAGOGUES': THREE CASE STUDIES

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The 'mini-interview ' is a new tool for evaluation of mathematics at the primary level. It is characterized by a short sequence of questions and tasks that a teacher can use with each of the pupils in the classroom to evaluate their level of understanding of a given concept, and also to gather some feedback about his or her teaching. But not all teachers are willing to use such a tool. We thus asked three 'orthopédagogues' - these are specialists who help teachers in their work and handle children with learning difficulties- to perform these interviews with regular third graders. Preliminary results of this experiment indicate that the three orthopédagogues found the time to integrate this component to their regular workload; the resulting evaluation proved useful to them as well as to the classroom teachers; it brought about a better understanding of the children's difficulties and thus about how to help them more efficiently.

The mini-interview is a new tool developed by Nantais for evaluating the understanding of mathematical concepts (Nantais, 1989, Nantais et al.,1983). As indicated by its name, the mini-interview is a short interview that a teacher can use with each one of his or her pupils and it can be carried out in the classroom with each child. It consists in a short dialogue between the teacher and the child within the context of a given clearly defined task pertaining to mathematics notions that have already been taught. The children are questioned about the task at hand in order to bring to light the procedures they are using. The correctness of the answer to a given problem or task is relatively of minor importance here, the emphasis being on their thinking as evidenced by their procedures.

The mini-interview has been tested (Nantais, 1987,1989) and the experiment has brought out the conditions under which a teacher can effectively use this tool: the training required, the classroom organization needed, etc. The results indicate that teachers can use this type of interview as a tool for formative evaluation, i.e. enabling them to gather some feedback about their teaching and useful information about their pupils' understanding of the mathematical concept at hand.

But not all teachers are ready to use such a tool for it is quite demanding: it requires a careful preparation, some changes in the usual classroom organization, time to analyze the pupils' answers, etc. This raises new questions: would it be possible and desirable to have someone else prepare the mini-interview, carry it out with each of the students and then analyze their answers? Who could perform this on a regular basis? What would such an operation bring to the teachers and their students?

THE 'ORTHOPÉDAGOGUES'

In a large number of schools in the Province of Québec, one finds a specialist called an "orthopédagogue" whose function is to work both with the children and their teachers.

With the children, the orthopédagogue takes care mainly, but not exclusively, of the diagnostic and remediation aspects: they work individually or in small groups (a maximum of four children) with those detected as having some learning difficulties, especially in French and mathematics. In some cases, "re-education" is required in order to help the pupils in developing cognitive structures that have remained inadequate. In other cases, their lessons will deal with a specific notion so that the child can link it to previously acquired knowledge, he or she can then re-integrate the class and learn more autonomously. Finally, in the simplest of cases, their intervention would simply be construed as pedagogical reinforcement, i.e. providing additional learning activities to children who do not have serious difficulties but whose learning is rather slow.

In order to be of assistance to the teachers, the orthopédagogues prepare and correct different written tests assessing the pupils' knowledge and abilities. They thus provide the teacher with an overview of their class indicating what the students have retained from their instruction. Considering the results of the assessment, the orthopédagogue might then suggest some appropriate learning activities to the teachers, or even help them build some.

RESEARCH QUESTIONS

Considering the nature of the orthopédagogues' work, one can perceive the usefulness that the mini-interview could have for such specialists: it would enable them to perform more complete and more formative evaluations, thus helping to detect the children's difficulties in mathematics and to plan some necessary remediation. It would also enable them to provide the teacher with a more detailed and more precise profile of the class and hence to plan more relevant and more rewarding learning activities. But obviously, all these questions at hypothetical and need to be verified, whence the following research questions:

- How can the mini-interview be integrated in the orthopédagogue's work?
- What would the orthopédagogue's use of such a tool bring to the teachers and their pupils?

METHOD

The best way to find answers to these questions is to experiment with real orthopédagogues working in normal conditions within the school context; hence our choice to proceed using case studies.

Three orthopédagogues, three women working in different schools, volunteered for the experiment, and where trained during a half-day session. During that period they were introduced to the mini-interview both at a theoretical and practical level. Following a short discussion on the general topic of evaluation, we presented them with the mini-interview and explained the spirit in which it was conceived, its objectives, how questions and material need to be prepared, and the way children are to be interviewed in order to assess their thinking in a reasonable amount of time. They were then provided with a prepared mini-interview (a questionnaire) on the concept of numeration. They were also given a framework in which to analyze the pupils' answers. This framework was based on a description of understanding the concept of numeration according to the Herscovics & Bergeron (1988) model of understanding. Finally, we explained what our experiment was about and what they would have to do to provide us with the data needed for our analysis.

We gave our volunteers a few weeks to get ready, during which time we remained available to answer their questions. Following this period of preparation, the three orthopédagogues used the mini-interview with each child in a Grade 3 class of their respective schools (n = 25).

levels, from kindergarten to Grade 6. Hence the orthopédagogues had to reorganize their work and do a few hours of overtime. Part of this time was needed due to the experimental nature of the work, especially the time spent on the final report and one of the two hours spent with the teacher. However, part of the remaining time used with each class would have been used under regular circumstances: the standard written test would have been given and corrected, the teacher would have been met, and work with some of the children would have been performed. Hence, the real extra time required by the mini-interview does not exceed 8 to 10 hours.

The orthopédagogues thought that this was quite a heavy load but they still found it acceptable for several reasons: because they did not have to perform such interviews every week in each of the classes, and more importantly, because they thought they would need less time with more experience in mathematics! This proved to be a great surprise, for all three admitted that, like a large number of their colleagues, they seldom worked in mathematics, preferring to work on the children's French, a field in which they felt more adequately prepared. In mathematics, they usually had been happy to administer some exams, report the results to the teacher, without doing much work with the children. In fact, the mini-interview had given them an opportunity to work in an area they had somewhat neglected. Thus, they did not consider the matter of time as a problem and believed they could reduce some of the time allocated to French and use it in mathematics, does achieving a better balance between the two disciplines.

They were full of praise for the mini-interview, and this explains their willingness to re-organize their work. They could only see the advantages in the new tool:

- for the children who, by trying to answer questions dealing essentially with their thinking processes, are brought to communicate their comprehension explicitly and thereby manage to gain a deeper understanding;
- for themselves, for they had found a tool enabling them to assess the mathematical processes used by all the children in order to uncover their difficulties, even in cases where these were concealed by acceptable performances on written exams. One orthopédagogue even

Each orthopédagogue then analyzed the pupils' answers in order to assess their understanding. This was then compared with another evaluation based on traditional tests written by the same children during the same period of time. Finally, each orthopédagogue communicated her conclusions to the respective teacher and used them in her own work with the pupils in the following weeks.

TYPE OF DATA COLLECTED

The data we gathered were of different kinds. It included:

- the tape recording of the interviews carried out by the orthopédagogues; the orthopédagogues' analysis of the children's answers;
- daily notes written by each orthopédagogue in a journal describing what happened and what they had done; this included remarks dealing with the mini-interview, the problems that had occurred, the solutions they had found for these problems, etc.;
- the orthopédagogues' answers to a questionnaire regarding the usefulness of the mini-interview in their work, about its advantages and disadvantages, about the changes they had to make in their regular work, etc.;
- the answers that the involved teachers gave to a questionnaire seeking their opinions about the mini-interview and the possible effects its use by the orthopédagogues might have had on their pupils;
- the results obtained by these students on the usual class exams;

PRELIMINARY RESULTS

In order to assess if the mini-interview could be integrated in the orthopédagogues' work, we first examined how much time they had spent on the experiment: an average of 20 hours! By and large, the interviews required between 10 and 15 minutes per student, thus a total of about 6 hours. To this must be added time for a short term preparation (?), (2 h.), time for giving the traditional written test ((1 h.), time for analyzing the answers to the test and those gathered in the interviews, (2 h.), time for meeting the classroom teacher (2 h.), time to work with the students afterwards (4 h.), and time to prepare their report on the experiment (3 h.). This amounts to a great amount of time allotted to a single class since the orthopédagogue must work with with each class at all stated: "It is a very convenient tool for finding those pupils who have developed mechanical routines without any concrete representation of the processes involved...something that traditional testing would not reveal". Let us add here that their analysis of the children's understanding seems to be, at a first glance, remarkably perceptive;

- for the teacher, since the orthopédagogues thought that with this new form of evaluation they could help them focus their teaching on the concepts involved and not just stress performance and good answers. Moreover, disclosing the strengths and weaknesses of a group at a given level would allow for some preventive interventions with teachers of the lower grades;
- for the parents, with whom communication becomes easier since the orthopédagogues are now in a position to provide clearer explanations about a children's mathematical knowledge and how they can be helped, if needed.

The orthopédagogues' enthusiasm is shared by the teachers whose classes were involved in the experiment. They contend that they were not disturbed by the interviews even if at all times, children were going in and out of the classroom to meet the orthopédagogue. In general, the conclusions drawn from the mini-interviews confirm much of their own evaluation but , for three or four students in each class, they acknowledge that the interviews revealed problems of understanding that could not have been detected through any of the written tasks. The three teachers liked the material that had been used in the interviews (chips for units , envelopes containing chips for the tens) and used it afterwards in the classroom. They also suggested their use to the parents of the children who had some difficulties. The teachers also stated that next year they would like to participate again in such an experiment and suggested that it be carried out at the beginning of the school year, so that right from the start, their might better direct their efforts towards the needs of their pupils.

BY WAY OF CONCLUSION

Despite the preliminary nature of our analysis, these results reflect the relevance of the mini-interview. The kind of renewal it brings to the process of evaluation seems to answer some real need. For the orthopédagogues, who up to then had done little in mathematics education despite the fact that it was considered part of their work, their responses are quite revealing. And so is the response of the teachers who , for instance, recognize that they have gained a better grasp of some of their students' understanding and who would be pleased to see the experiment tried again some time in the future. The fact that the mini-interviews were carried out by someone other than the classroom teacher does not seem to have affected the effectiveness of this new tool. And with this effectiveness now acknowledged by the teachers, perhaps it will be possible to convince them to handle some of the interviewing themselves or to participate more actively in their realization.

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DEVELOPING PROBABILITY AND STATISTICS FROM PROBLEM SITUATIONS: AN EXPERIMENTAL COURSE FOR PROSPECTIVE TEACHERS

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This paper reports findings of a participant observer in an experimental class in probability and statistics for prospective elementary and middle school teachers. Based on a constructivist approach, the class introduced concepts in an integrative fashion through problem situations. The researcher found evidence of students adopting an approach toward problem solving of attempting to make sense of the situation.

NCTM (1989) has created a vision about the teaching of mathematics that includes the learning of mathematics as an active, constructive process with instruction based on problem situations. This paper reports results of a participant observer at a large university in a class in probability and statistics for prospective elementary teachers (K-8) that conformed to this vision. Data were fieldnotes and audio-recordings of interviews.

PORTRAIT OF A CLASSROOM

Mathematics instruction was based on the LES Model of the Middle Grades Mathematics Project of MSU (Shroyer, 1984). A problem was posed, and students made conjectures, with justification, about the outcome. The activity was then explored, and during debriefing, concepts were developed jointly by teacher and students through dialogue. Concepts were presented in an integrated way with each new concept appearing first as part of a problem situation. Students explored aspects of the concept in different activities and over time developed an understanding of the entire concept. The language used was informal, with terms, such as expected value, used in context until they became part of the working vocabulary of students.

MATHEMATICS AS A CONSTRUCTIVE ACTIVITY

This teaching scheme was a deliberate attempt to help students adopt an approach of "making sense" of probability and statistics. In an interview, the teacher explained she viewed mathematical knowledge as nodes of structurally related ideas, such that knowledge added to one part affects many other parts, which in turn must be adjusted. Concepts are never fully formed, but evolve as students construct new or additional meaning. The teacher is questioner, tasksetter, and guide for assisting students to construct a coherent realization of a concept. Starting from problems is intended to facilitate students' reflecting, questioning, and reevaluating previous ideas.

This teaching style is consistent with a constructivist approach to learning as defined by von Glasersfeld (1987). He characterizes knowledge and competence as the product of students' conceptual organization of their individual experiences, not a transferable commodity that can be conveyed by communication. Learning is drawing conclusions from experience and acting accordingly, and making sense is organizing experiences in some way or with a view to making predictions about experiences that are to come.

This teacher's approach is in marked contrast to what occurs in typical classrooms, where a few concrete or case specific examples of a concept are compared in order to extract elements common to them all and thereby abstract the concept itself. Discussion then moves to formal, simplified versions of the concept with applications following to produce the desired "understanding". Such schemes assume that if one understands the structures in different symbolic representations and if one perceives the isomorphism between those representations, then one can abstract the underlying mathematical concept (Schoenfeld, 1987). Such an approach could obscure facets of a concept, thereby preventing a full understanding of it (Borasi, 1984).

Teaching that focuses too directly on abstracting formal concepts does not account for the complexity of concept understanding. Tall and Vinner (1981) distinguish a concept definition and a concept image. They define concept image as:

> The total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. [...] All mental attributes associated with a concept, whether they be conscious or unconscious, should be included in the concept image.

The concept definition must contain only those properties common to all instances considered, but the concept image very likely contains other elements, which pertain to specific instances that may be evoked when dealing with the concept. Students in the process of acquiring a preliminary understanding of a concept may have an incoherent concept image without being aware of it.

Janvier (1987), also separating concepts from conceptions, suggests that the main questions regarding development of conceptions concern timing. When meaning for students appears to vanish as soon as they are presented with formulas, it may be because the concepts themselves are so powerful. Thus, he advocates understanding being developed at the preconcept stage prior to learning the formal concept.

This teacher's holistic approach that continued to focus on meaning in the problem situation, rather than relying on abstracting commonalities that could result in students' dismissal of meaning prior to some reasonable level of understanding, is consistent with Borasi, Janvier, Tall, Vinner, and Schoenfeld.

RESULTS

What evidence is there that students developed a view of learning mathematics as a complicated, constructive activity? In one interview when asked if this course had changed her thinking about mathematics, one student replied:

> Yeah, I wanted to do it more straight forward, but I see, the more open you are to different ideas, and allowing the kids to do examples and experiments themselves allows them to understand it more better. [...] If they don't get hands-on experiments, they are gonna try to understand, but like me, I had problems somewhat myself, but when I was in her class, and we did hands on, I could remember it, and do things. I could remember it. I could grasp the concepts; it was so much easier. I didn't have to work at it. It was there!

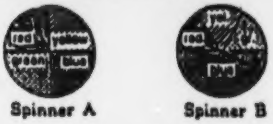
This same student said she took this class so she could teach-- so she could understand. When questioned further, she said:

> It's how she presents things, in the class-- allows you to be able to usually pick it up from one thing and say that this relates to this other thing. So it's not that difficult; you just gotta relate things.

Another student made connections between classes and between teaching and learning. She risked writing a paper for this class on a topic from another class that she did not understand. She successfully critiqued a journal article on regression toward the mean and discussed its relation to teaching the topic even though the necessary sophistication level seemed inconsistent with her class performance. Later, she proudly stated she now understood the topic. She learned that one can learn by trying to teach, and her topic choice suggests she was motivated by wanting to 'make sense' of it.

Written work on tests provides evidence of sensemaking activity. Figure 1 is a problem from the first test.

2. Another carnival game is called "Making Purple". It is played on the two spinners pictured below. The object is to spin twice and land on the colors needed to make purple (red and blue). The choice of spinners is up to you.



Patty chose to spin twice on spinner A. John chose to spin twice on spinner B. Mary spun first on spinner A then on spinner B. Who has the best chances of making purple? Explain in detail how you reached your decision.

Figure 1.

(Identify the three problems as AA, BB, and AB for the two spinners used.)

Problem 2 was an extension of previous problems. Any prior spinner problem involved spins on the same spinner, except for a single problem in the context of binomial distributions that involved 6 spins on the same spinner, and which was solved using Pascal's triangle.

Students could not rely on a formal algorithm unless they categorized each part of this problem as involving a sequence of two independent events and also as involving two mutually exclusive cases within each problem. The work of three of the four students did indeed illustrate that they were guided by this method, but each failed to carry out the process completely. Each altered the process for the problem AB with two of the them adding, instead of multiplying, probabilities. This alteration suggests that as the complexity of the problem increased, the students either worked harder on it, or they were influenced by the representation.

When asked why she had added in the case AB, one student referred directly to the representation of two separate spinners:

> I was thinking while adding, I got this problem, and this problem. Now all I have to do is to add these together to get the third probability because they are separate. In my mind I made them separate, really separate. When I pulled them together, they weren't being multiplied, because they were so separate. When it was in its own wheel, you know, it's altogether, and I'd multiply. You do the regular thing, but when they are so separate I couldn't pull them together.

Her reference to "do the regular thing" indicated she used an algorithm of multiplying two probabilities for two sequential events, yet was strongly influenced by the representation to make sense of the problem in some other way.

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In addition, her work displayed a considerable amount of other activity, implying she was actively trying to make sense of the situation, rather than applying a well-learned algorithm. In one corner was an area grid for geometrically computing probabilities. Other calculations, which suggested she was trying to compute expected value, lead the researcher to believe (as was later confirmed) she was linking this problem to one from class that involved sequential, independent events.

An area grid would have provided a simple, accurate method for all three parts of this problem. When asked why she had drawn, but not used, the grid, she explained:

> I had, but I erased everything before. If you looked underneath it, it would all be erased because it didn't make sense to me; it wasn't working. The answers weren't working with my logic. I was saying there's no way that this could be right. One of them has to be more than the others. There couldn't be a tie or anything like that. So I erased what I had underneath there. [The probability of getting purple from both BB and AB is 1/6.]

She explained that it didn't seem logical that the probabilities could be so inconsistent with the proportions between spinners.

The fourth student's work implies that she tried to make sense of the problem as a problematic situation, rather than to identify a problem type and apply an algorithm for solution. She used none of the previous methods, but created a probability tree for each of the problems. Her use of trees was surprising for two reasons. While students had used trees to create lists of possible outcomes, prior to this test, probabilities were listed on the branches only in the context of a maze problem in which an area grid was the primary vehicle of problem solution. Had the student considered this, she would likely have used the area grid since later interview responses confirmed she understood well the grid's use. Second, up to this time, she had never seen a tree with unequally likely branches. Yet to account for P(blue) = 1/2 on spinner B, while each other probability was 1/6, she simply added two branches for blue so the tree had six equally probable branches. Except for a minor computation error, she solved the problem correctly. When queried about her method, she replied that she had been introduced to the tree for creating lists in middle school and thought it was a useful tool. During the term, trees appeared in her written work, either as a solution tool or a check of a problem solution, indicating that she regularly made sense of problems in her own way, using her own devices in addition to any tools introduced in class.

CONCLUSIONS

In summary, individual students seemed to see the value of activities in constructing mathematical understanding and to try to "make sense" of the mathematics they were learning, rather than to categorize problems and apply algorithms. During the course of the term, conjecturing moved from single conjectures with reasons to debate over their relative merits. Students made connections to other mathematical experiences and to teaching.

There are, however, limitations to the conclusions that can be drawn from this small study. The class was self-selecting and, thus, attracted highly motivated students. The inter-action during each class between the teacher and each class member provided unusually rich social interaction that may not be possible in larger classes. It may, however, point to what is possible when such interaction takes place.

A more serious question is that of research method. The nature of instruction requires that research be contextual, yet it is difficult to document what mathematics is being learned. Interviewing and stimulated recall for written and class work needed to be supplemented with task-based interviews. Such interviews could provide useful information, yet they are an intervention in that they provide the very instruction that is being researched. If we are to study this kind of mathematical instruction, we need valid research methods, or at the very least, better ways of accounting for the intervention of the data collection on what is learned.

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CHANGES IN PRE- AND IN-SERVICE TEACHERS' VIEWS OF PRIORITIES IN ELEMENTARY MATHEMATICS AS A FUNCTION OF TRAINING

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This paper reports on two efforts to expand teachers' conceptions of the scope and goals of elementary mathematics education. The first study assessed the impact of a video-based workshop series on in-service teachers' ability to analyze children's mathematics performance and on their views about content and assessment techniques. The second study focused on changes in the quality and content of questions asked by students in a pre-service elementary mathematics methods class as a function of instruction that included problem-solving, hands-on approaches, and videotape analysis. The results of both studies indicated some shift toward more precise labeling of observations and educational concepts as well as some tenacity of beliefs. Differences between the needs and conceptual frameworks of the two groups are discussed in terms of their implications for professional development programs.

Contemporary views of elementary mathematics education have placed increasing emphasis on problem solving, active learning, and an expanded conception of the scope of mathematics curricula (National Council of Teachers of Mathematics, 1989). While few mathematics educators would argue with this emphasis, the reality is that most practicing teachers and current students preparing for careers in elementary education were personally educated at a time when elementary mathematics was conceived of as little more than the acquisition of basic skills and mastery of computational competence. Therefore, in calling for new ways of teaching children, we are asking teachers to embrace an orientation and utilize a knowledge base that may be alien to their own ways of thinking about mathematics (Clark & Peterson, 1986). In order to carry out the spirit and not just the form of the contemporary view of elementary mathematics, then, both in-service and pre-service teachers need to be reeducated in their own conceptions of elementary mathematics. The purpose of this paper is to report on two attempts to achieve this goal.

STUDY 1

Project Description. During the fall semester, a series of five workshops was run as part of an in-service teacher enhancement project conducted in an urban school district. The project, sponsored by the National Science Foundation (Ginsburg &

Kaplan, 1988), focused on developing elementary teachers' sensitivity to the variety of ways in which children understand and interpret mathematics content presented to them. During the workshops, conducted after a pre-intervention evaluation session, participants were asked to view and analyze videotapes of preschool through third grade children engaged in some mathematics tasks. The workshops were attended by about 75 volunteer teachers, staff developers, and supervisors. Following the final workshop, attending participants were asked to complete a post-experience evaluation form.

Evaluation Procedures. The participants were evaluated on two tasks intended to assess their conceptions of what was important in elementary mathematics. The first measure consisted of their open-ended responses to an observation of a short videotape segment of a first grade child engaged in some simple counting and arithmetic tasks. On this measure respondents were asked to report the "highlights" of what they observed on the tape. On a second measure, participants were asked to complete an open-ended self-report questionnaire including questions about what they thought were the most important things to be learned in elementary mathematics and how they thought these could be best assessed. Before and after responses were coded according to categories of behavior and concepts that were spontaneously reported.

Results. In the pretest situation there were a total of 28 respondents on the video task and 32 respondents to the questionnaire (a return rate of about one in three of those in attendance). On the posttest there were 15 respondents to the video task and 16 to the questionnaire (also a return rate of about one in three of those in attendance). Distinct responses within each individual's protocol were scored separately and summaries of the occurrence of response categories were calculated as percentages of the total number of responses made at pre-and post-assessment periods.

The most salient trends noted in the data were the following: In general there was a marked increase in the specificity of observations made on the videotape. Most notably, there was a decrease in the number of vague references to general ability or some affective state and an increase in the number of specific solution strategies spontaneously recognized. On the questionnaire item referring to the best methods for assessing learning, there was an increase in statements about particular individual interviewing techniques and a slight reduction on reliance on written testing procedures and unspecified forms of observation. There were few changes, however, on responses to the question regarding the most important things to be learned in elementary mathematics. Respondents tended to hold fast to (and even increase the mention of) vague notions about conceptual processes and unspecified problem-solving skills.

STUDY 2

Project Description. The second study took place within the context of a small pre-service methods course in elementary mathematics. Participants were seven post-baccalaureate students who attended eight class sessions, each for four hours. The focus of the course was to model problem-solving, hands-on approaches to contemporary topics in elementary mathematics and to increase personal understanding of mathematical concepts. Analysis of videotapes of children actively engaged in mathematics tasks was also included in the course.

Evaluation Procedures. The questions that the students wrote on a weekly basis served as the material through which changes in their conceptions about what is important or problematic in mathematics instruction could be inferred. Responses were categorized according to the concepts, procedures, and behaviors that were spontaneously mentioned.

Results. The number of questions asked in each of the categories was totaled for each week and a percentage of occurrence for each category was tabulated within each category. A comparison was made between the first four weeks and the second four weeks of the course. Analysis of the data indicated a continued predominance of interest and concern about teaching techniques and personal knowledge of mathematics over the eight-week course of study. However, a relative increase in the specificity of questions asked about teaching methods (including materials, procedures, and concepts) was noted. There was, for example, a decrease in questions such as, "Are there any creative ways to teach the skills other than by rote?" and an increase in questions such as "What type of manipulative would you prefer when teaching decimals? Dienes blocks or squared paper models--or would you choose something entirely different? Why?"

Unlike the in-service teachers, this group did not spontaneously raise issues about problem solving, interviewing techniques, or children's solution strategies. Only a very slight increase in concern about techniques for dealing with individual differences in learning was noted.

DISCUSSION AND CONCLUSIONS

Although the results of these studies are essentially qualitative and constrained by small numbers, they do point to some interesting trends. First they suggest that while some beliefs and concerns of in-service and pre-service teachers can be modified by the kind of training provided in these studies, others may remain unchanged. They also indicate that the concerns, and consequently the content that was learned, varied as a function of differences in the conceptual frameworks that participating groups brought to their experience.

For example, after training in the videotape workshops on what to look for and how to use interviewing techniques, in-service participants seem to have enriched their understanding of ways to analyze children's conceptual and procedural knowledge in mathematics. However, this new knowledge did not diminish their tendency to refer only to vague generalities about the importance of problem solving strategies. These findings are consistent with the Peterson, Fennema, Carpenter, and Loef (1989) position that teachers' beliefs, thoughts, judgments, and knowledge affect the extent to which they implement a training procedure as intended by the procedure developers. The findings also support the position articulated by Adams (1989) that thinking skills introduced and developed in some specific context, will be remembered, understood and accessible only in relation to that context.

Second, in contrast to the in-service teachers, the pre-service teachers did not make the same kinds of knowledge gains. Although exposed to the same videotapes as the in-service staff, they did not come away with an increased appreciation of children's strategies and potential skill for conducting individual assessments. Rather, their attention remained drawn to the practical concerns of a beginning teacher, i.e., what the teacher should do and how could it best be done.

The differences observed in what the in-service and pre-service groups learned and how their concerns changed or remained fixed, suggest that it is not enough simply to provide "models" of instruction as some suggest (Hyde, 1989) nor is it enough even to provide information with an opportunity to practice (Joyce & Showers, 1988). Rather it is up to us as teacher educators to acknowledge the varied perspectives of in- and pre-service teachers and then adapt our instruction to their vantage points. In taking their perspectives into account, we increase the probability that the message of contemporary mathematics educators will be heard and put into practice.

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CONSTRUCTIVISM: A MODEL FOR RELEARNING MATHEMATICS

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A large number of students enter college each year not knowing basic mathematical concepts such as fractions, decimals and percents. Although many students take mathematics for twelve years in elementary school, junior high school and high school they lack the fundamental processes necessary to be successful in college level mathematics courses. In this paper a teaching model is outlined that will benefit those students in the learning of mathematics and mathematical processes. This model is based on the tenets of constructivism as put forth by ´von Glasersfeld (1983).

Since January of this year, the public and the educational community have been deluged with report after report (Romberg, 1989; National Research Council, 1989; etc.) of the dismal state of mathematics education in this country. Although this may appear to be a new phenomena, it has long been seen in college developmental mathematics classrooms. Twenty-five percent of all incoming college freshmen are enrolled in developmental mathematics courses (arithmetic, elementary algebra and intermediate algebra), with eighty-five percent of colleges and universities across the country offering such courses (Hall, 1985). In addition, Lawson and Renner (1979) found that more than fifty percent of all college freshmen tested with Piagetian-type measures were still at the concrete operation level of thought. Not only are students not learning the mathematical content that is being presented in the schools, but more importantly, after twelve years of mathematics, they have not learned how to think, or process mathematically. College developmental mathematics programs try to teach all the mathematics that the students have not learned in the pre-college curriculum in the shortest possible time with little thought given to teaching the students to think mathematically. The traditional methods of instruction are the lecture approach and the programmed study approach (McDonald,1988). However, high failure rates indicate that these methods are not appropriate for most developmental college students (Garfield, 1988). This should not be surprising since many of those failing have built up twelve years of misconceptions and procedural bugs (Brown and Burton, 1978). Procedural bugs are errors in a student's algorithms that are consistent and systematic. These approaches may foster only more misconceptions and procedural bugs in that the students are not given "the opportunity to experiment directly with a reality that contradicts their beliefs" (Nesher, 1988; p 71).

A STUDY OF DEVELOPMENTAL COLLEGE MATHEMATICS STUDENTS

In fall, 1988, twenty-five students who enrolled in a developmental arithmetic course at a large urban university were interviewed and asked to provide self-reports of their experiences that related to the learning of mathematics. The questions were open-ended and ranged from specific information to broad questions relating to feelings and attitudes toward themselves as learners of mathematics and mathematics in general. The results are summarized for questions which asked the students to reflect, in detail, their previous experiences in mathematics, relating to both classroom learning and out of school mathematical experiences. The students reported that mathematical knowledge and sophistication gained during the early years was limited and fragmented. Furthermore, all subjects received less than 80 percent on an arithmetic pretest, a test which consisted of computations and word problems involving whole numbers, fractions, decimals and percents. At the same time, more than 85 percent of the students reported that they had taken geometry and two years of algebra in high school. Throughout their mathematical career, it was evident that the teacher was the center of the universe, giving out the knowledge and providing the motivation for learning. When the student felt any satisfaction or sense of achievement, the teacher was there to reward or to withhold rewards, thereby providing the student with evidence that the learning of mathematics was out of the student's control. Seventy-two percent of the students reported that by the sixth grade, they knew that they were poor mathematics students. More than fifty percent indicated that their downfall was learning the multiplication tables. Ninety-two percent related vivid memories of being embarrassed or humiliated by a teacher in a mathematics class during their pre-college schooling.

Throughout junior high and high school, as the question about knowing mathematics again surfaced, as is natural during the search for identity during the teen years, many students learned that the mathematical world was not what it seemed. The lack of understanding that was evident in earlier years, reared its ugly head as the students were thrown to the "algebra" wolves. The eight percent of the students who reported being successful in the elementary school became lost by ninth grade algebra, implying that prior to high school eighty percent of these students had already developed poor self-images of themselves as mathematics learners, yet they most of them continued to take mathematics in high school. Not one student reported any positive interventions on the part of teachers, guidance counselors or parents. Through these years their mathematical knowledge remained stagnate. Mathematical representations of the world were only partially developed. Each student developed his or her own incompatible view of the world and his or her own mathematics (misconceptions and procedural bugs). And, to make matters worse, mathematical experiences were limited. Since mathematics only involved calculations, calculators took away any rationale for learning mathematics. The student knew that the mathematical world (i.e. the mathematics classroom) was not the world in which he or she wanted to live, but knew that he or she would have to cope. By this time, the world in which the student lived had become a world of "objective facts" or "things-in-themselves" given by others, to the student. At no time was the student involved in the development of his or her own mathematical knowledge. Over eighty percent of the students recalled sitting in high school mathematics classes (algebra, geometry and basic mathematics) feeling lost and unable to grasp the material being presented. When asked why they enrolled in yet another mathematics class in their first quarter of college, ninety-two percent said that it was required, and that they had no choice. Of this ninety-two percent, twenty percent said that they would have taken mathematics anyway, especially since it would be a chance to "start over". Although these results are preliminary, they do suggest that for a certain group of students pre-college mathematical experiences provide little mathematical understanding and negative attitudes with respect to mathematics. If the students select to take mathematics in college, as required by choice of major, or specific collegiate requirements, then it is necessary to consider instruction that attempts to overcome negative experiences, detect and address misconceptions and procedural bugs, and provide an environment in which students can begin to understand the mathematical world. One such instructional model is one based on the tenets of constructivism.

APPLYING CONSTRUCTIVISM TO THE COLLEGE CLASSROOM

Von Glasersfeld (1983) contends that "knowledge is not a transferable commodity" and the teacher cannot be the dispenser of "truth" but a facilitator in conceptual understanding, "it is the student who must do the conceptualizing and the operating." Furthermore, von Glasersfeld suggests that "if students are to taste something of the mathematician's satisfaction in doing mathematics, they cannot be expected to find it in whatever rewards they might be given for their performance but only through becoming aware of the neatness of fit they have achieved in their own conceptual construction" pp 67-68. This view of learning is referred to as constructivism. Constructivism calls for both the student and the teacher to be learners. The teacher must first construct his or her knowledge of the student's present cognitive structures. This can be done through observations and interviews with the student. Once this is accomplished, the teacher's role is to modify those structures to become like the "adult" cognitive structures. This is achieved by means of "indirect guidance". As learning is not only for the student, but also the teacher, the teacher must develop a model of the student's conceptualization, and assess possible avenues for that student. This can only be done if there is "real" communication between the student and the teacher. The teacher must establish the framework for this communication through a common language. Fulwiler (1982) states that language is the symbolic system in which we receive, transmit, and process information as well as represent, study, and understand the world. Furthermore, Fulwiler reports:

> We think things by talking to ourselves, carrying on "inner" conversations in which we consider, debate, and rationalize. The key to knowing and understanding lies in our ability to manipulate internally information and ideas received piecemeal from external sources and to give them coherent verbal shape. We learn by processing, and we process by talking -- to ourselves and to others (p.17).

Piaget believed that language is the basis for scientific reasoning and as an individual moves from the concrete operational stage to the formal operational stage (abstract thought), sensory and perceptual experiences give way to symbolic representations. This process can be enhanced through the use of language, both written and spoken, as put forth by Bruner (1966) and Lesh (1979). Britton (1970) extends the idea of speech when he suggests that if we are to make sense of reality, we must be able to symbolize reality, and we do this through speech. In order to come to an understanding of a concept or event, we talk to others about the concept or event. The purpose of talking is not just to communicate but to internalize that concept or event.

According to the constructivist point of view, a teacher must continually communicate with the student and through this communication comes the teaching, and hence the learning. According to Cobb and Steffe (1983) a successful teaching communication will occur "whenever the teacher's actions are guided by explicit models of the children's mathematical realities. From this perspective, the activity of teaching involves a dialectic model between modeling and practice" p.86. The students' constructions are generated by the impact of these interactions on the student's own cognitive structures and not the interaction itself.

TEACHING MODEL

A model of instruction for the relearning of mathematics based on a constructivist view point needs to provide the teacher and the student ample time for interaction and "indirect guidance". This can be achieved only when the teacher is taken out of the role of the source giver of knowledge and makes the student the center of the learning process. The use of pair-problem solving (Lochhead and Whimbey, 1982) and/or small cooperative groups are possible alternatives that could allow the teacher to interact with individual students. In addition, well developed questions and activities would provide the pairs or small groups the experiences that are necessary for them to internalize the thought processes that are required to learn mathematics. The teacher is continually interacting with either individual students or several students, determining each student's state of mathematical understanding. As the teacher interacts with each student, the teacher is able to modify that student's present concepts and operations. While the teacher is working with one student or several students, the rest of the class, whether in pairs or in small groups, is testing or practicing their own hypotheses or ideas or processes in a non-competitive and non-judgmental environment. The use of pairs allows the students to practice questioning and monitoring. The teacher can also use writing as another tool to help assess mathematical understanding. When students are asked to write out a protocol (procedures used to solve a particular problem), the teacher is able to assess concepts and operations. It is also a mode in which a student can monitor his or her own processes. Writing allows a student to review and revise.

Once a protocol is written, a student can orally follow the protocol and attempt to follow the logic of the protocol. If the logic appears faulty, the student can correct it, or question the teacher. In turn, the teacher can, in writing, question the student. By keeping a detailed journal of problem protocols, a student can "track" past errors and areas of difficulty. Furthermore, writing can serve as a catharsis for students with math anxieties by expressing their fears, dislikes, frustrations and anxieties, as well as their successes and achievements. Through the oral and written interactions, the student is able to reflect on and attempt to resolve any cognitive conflict, which is necessary if the student is to going to come to "know" mathematics.

College developmental mathematics programs cannot change the students' past experiences, but can, and should, use those experiences to foster and encourage new mathematical growth.

For students who have spent twelve years sitting in mathematics classes, and are still lacking basic mathematics concepts and processes, relearning mathematics through a constructivist approach may provide the impetus that is needed. The constructivist approach would: 1. provide students the opportunity to demonstrate many of their misconceptions and procedural bugs and then be given the opportunity to be carefully confronted with those misconceptions and algorithms, thus creating the need for cognitive conflict" (Bruner, 1966) that is necessary for cognitive development; 2. encourage students to become the center of their own learning by offering opportunities that allow the student to build on his or her own views of the world; and 3. gain the confidence and experience the satisfaction that can only be achieved when students develop internal curiosity and motivation.

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EXAMINING CHANGE IN TEACHERS' THINKING THROUGH COLLABORATIVE RESEARCH.

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Eight teachers from grades 7 through 10 were involved in phase one of a three year study examining the effects of collaborative research on change. Indications are that parallels exist between proven methodologies facilitating student learning and the collaborative method of professional development.

This paper describes the initial phase of a project designed to examine change through a co-operative networking system comprising a symbiotic relationship among classroom teachers, a mathematics consultant and two teacher educators. One focus of inquiry was to examine changes in teacher thinking resulting from participation in the project.

Although there has been a change in the content of middle school mathematics programs over the years, there has been little change in teaching style. Teacher explanations dominate most classrooms, with teachers being more concerned with classroom control than mathematical meaning, and mathematics being viewed as a collection of facts to be absorbed rather than as a collection of ideas to be examined and discussed. (Research Advisory Committee, 1988, p.341; Romberg & Carpenter, 1986, p.851; Lapointe, Mead & Phillips, 1989).

It has been suggested that it is necessary for teachers to recognize the benefits of new ideas if modifications to teaching style are to be realized. (Howson, Keitel and Kirkpatrick, 1981, p.8). The more traditional forms of information dissemination, such as workshops, presentations and printed materials may not provide enough opportunity for teachers to discuss innovations and internalize new ideas for themselves. Moreover, transmission of information in this traditional style may not be the most effective means of assisting teachers in their development of new conceptual frameworks. We know from research with children that what is learned is seldom an exact replication of what has been read or heard (Brandt, 1988/89, p.15). Why should this be different for teachers?

This project was based on the belief that teachers do need to reflect on both theory and practice, and consequently it provided an opportunity for them to critically scrutinize their own teaching within a supportive environment. Such reflection by teachers on their own practice is an essential component of professional development since such reflection allows them to identify research findings which they consider beneficial (Campbell, 1988, p.102), and a rationale for change becomes explicit.

Diagnostic teaching, incorporating conflict discussion (Bell et al., 1985), was the strategy used in the classroom. Although students are more likely to construct correct conceptual frameworks when taught in this fashion rather than in the more traditional expository style, communicating this philosophy through print is inadequate (Onslow,1986).

The reactions of teachers to a change in emphasis in their teaching style is examined by addressing the following questions:

1. What changes in teachers' thinking, concerning the learning of mathematics, took place during project participation?
2. Which project elements appear beneficial in effecting change?
3. What support appears necessary to facilitate change?

METHOD

Eight volunteer teachers (two from each of grades 7 through 10) participated in the initial phase of the project, which involved deliberations about the philosophy underlying conflict discussion in the mathematics classroom; the teaching, by pilot teachers, of select lessons, all of which were observed and some video taped; the design of materials to accommodate specific teaching situations; and follow-up meetings to offer support and to discuss strengths and deficiencies of the teaching strategy.

Prior to the commencement of the project, each of the eight teachers completed a questionnaire designed to clarify their philosophy of mathematics education, and describe their allocation of time to various teaching and assessment processes. Throughout the year teachers were encouraged to record reflections on teaching style and developmental process in journals, providing a chronological overview of teachers' attitudes towards the process. At the conclusion of year one, teachers were individually interviewed to reassess their philosophy on the teaching of mathematics and to discuss changes which they or the research team had observed.

Students in participating classes wrote a general diagnostic test in the initial stage of the project. In addition to diagnosing students' difficulties the process was deemed necessary so that the seriousness of researchers' concerns regarding students' conceptual obstacles could be recognised and personalized by the participating teachers.

The teaching style was designed to promote conflict, allowing children to wrestle with difficult concepts and provide justifications for their conclusions (for a detailed description see Bell et al. 1985; Onslow 1988).

ANALYSIS OF THE DATA

The original three questions are examined using data from questionnaires, journals, observations and interviews.

1. What changes in teachers' thinking, concerning the learning of mathematics, took place during project participation?

Before the project commenced, teachers were asked what percentage of time, over the course of a year/semester, they would devote to each of the following activities (see Table 1).

Table 1. Percentage of Time Spent on Teaching Activities

Teaching Activities	Time(%)							
	Grades 7 & 8 teachers				Grades 9 & 10 teachers			
a) small group work	50	10	5	35	5	20	0	0
b) class discussion	15	5	10	10	5	20	5	5
c) give suitable examples followed by practice	25	60	60	10	45	20	75	75
d) diagnostic assessment (prior to teaching)	5	0	0	5	5	1	0	0
e) summative assessment	5	20	15	15	25	15	15	15
f) discovery work	0	5	10	25	10	15	5	5
g) other	0	0	0	0	5	9	0	0

Individual results indicated that we did not have a homogeneous group and teachers had very different views as to how students should spend time in the mathematics classroom. The picture is brighter than, but not too different from, that found in other recent surveys in which teacher explanations followed by practice in textbooks tends to be the norm (Research Advisory Committee, 1988, p.341; Lapointe, Mead & Phillips, 1989, p.21).

Interviews at the end of the first year/semester provided information on how teachers perceived changes in their teaching:

> I didn't see this kind of thing (conflict discussion) relating to math. I use it in history. Now I look for the opportunities in math to have kids justify why they have a specific answer.
>
> Discussions were lengthy but good because I could see what the children were doing. It was beneficial. We got into arguments and explanations and in math we never really do that. In math it's like, here's the question, here's how to do it, now go ahead and do it.
>
> I would not have believed students would have so many difficulties with certain concepts if I had not participated in the project. It opened my eyes that we sometimes assume too much. That was a let down for me. Something I perhaps didn't want to know.

It should be noted that whereas changes in teacher thinking became evident during the course of the project, changes in teaching style were less distinct. Observations indicated that although discussion and analysis was taking place in the classrooms, teachers had difficulty changing their role from transmitters of information to chairpersons of conflict discussions, and consequently remained the dominant persona in several classrooms, with most class discussions being between teacher and student rather than between student and student.

There was also evidence of teachers formulating an explicit awareness of implicit actions already employed in their teaching:

> I think I was doing some of this before, but I was not totally aware of why.
>
> Indirectly, I think I have been using a similar method but hadn't realized it, and hadn't taken the kids as far as I now know is possible.

Becoming aware of why implicit actions produce worthwhile learning activities allows teachers to incorporate such actions more frequently and confidently into their programs.

The main disadvantages of this teaching style from the teachers' perspective, especially at the secondary level, were the overcrowded curriculum and the shortage of time:

> I think my class would benefit from this approach. They would understand the material better, but they would suffer because they wouldn't finish the program.

Such comments substantiate the fact that teachers are wrestling with a dilemma, and need assistance in removing unnecessary curriculum, if change is to become a reality in the secondary classroom.

2. Which project elements appear beneficial in effecting change?

Four major elements appeared to be important in effecting change during the initial phase of the study.

a) Teachers said that having their opinions valued and respected by others in the group made them feel more professional. Constructive criticism was evident at all meetings, and teachers found this support beneficial:

> It is important to have your opinions valued - it makes you feel that you are a professional.

> I appreciated the open and supportive atmosphere. I think this is critical to this kind of project.

b) An environment must be created in which teachers feel willing and able to take risks if we are to see growth and meaningful change:

> My comfort level increased as the project progressed. We didn't know what we were doing in the beginning. I can admit that now.

> We've been able to try something, come together, and say this is what I did and this didn't work. Somebody else was able to say well I did this a little differently and it was great.

Not all teachers felt free to take risks. One participant presumed there was a "hidden agenda" (see Campbell, 1988, p.107). As a new teacher, this person felt uncomfortable with the project:

> I mean in some ways all of us were guinea pigs with the method and I was the guinea pig within the group.

Although the project leaders became somewhat aware of these feelings part way through the project, reassurance to the contrary did not alleviate the anxiety. As expected, this teacher did not wish to continue for a second year.

c) Teachers need to believe in what they are doing and that the change will benefit their students' education. Teachers must arrive at these beliefs independently; we can no more tell them what is fact than they can tell their students what is fact, and expect success. The general diagnostic test allowed teachers to recognise and personalize their students conceptual difficulties for themselves. Class discussions provided further occasions for teachers to recognise that while their students did not understand some basic concepts studied over previous years, they did understand other concepts only cursorily covered:

> There were kids that were missing things that...gee, I just took for granted that they had. Some things were coming out that they didn't understand, some very basic things that I assumed they already had.

d) Reflection is possible in isolation, but more difficult. Teachers found it valuable to reflect and discuss ideas amongst themselves and to have a mix of elementary and secondary teachers involved in the project. Although the elementary teachers felt somewhat anxious about the content, they were more comfortable than the secondary teachers with the pedagogical style. Interaction raised the comfort level in both areas:

> It wasn't one shot in the dark, and it wasn't someone just presenting a bunch of ideas to you. It was a sharing of ideas and people talking about their experiences in relation to their ideas.

> I got a lot from everybody else, especially the elementary teachers. I got a whole different perspective on how elementary students think.

3. What support appears necessary to facilitate change?

In addition to developing those elements described in the previous section, the importance of involving principals and/or department heads became apparent:

> We need to get the principals involved. If the principal is doubtful about the program then it's useless for me to be in a school where I'm not getting internal support.

Although principals and department heads were invited to join the meetings, the importance of their involvement was not stressed and only one administrator attended a meeting.

During the year a parallel between the strategy for teaching students and the developmental process for teachers became evident. The climate of trust and mutual respect advocated for classroom discussions was just as crucial for teachers' meetings. Time to develop conceptual frameworks was necessary in both instances. Several teachers also made the comparison:

> It's just like the kids. If you give me something to read, I may not read it. If you tell me, I'm likely to forget it. But if you let me do it, I'll have a chance of understanding, especially if I have support.

CONCLUSIONS

The image of the teacher as a reflective professional (Clark and Peterson, 1986) attempting to make explicit sense of implicit theories and beliefs concerning learners and curriculum is supported by this study. Teachers are often isolated, however, seldom sharing ideas about children's mathematical understanding or pedagogical techniques with their peers.

If being aware of what students do understand rather than what is assumed to be understood is a pre-requisite to preparing meaningful classroom experiences, then teachers must be given the opportunity to analyze and discuss the difficulties inherent in learning mathematics (Carpenter, Fennema, Peterson and Carey, 1988.) Teachers also need to practice and reflect on processes beneficial to meaningful understanding. Without such provisions, transmission of knowledge from researcher to teacher is probably no more effective than transmission of knowledge from teacher to child.

It appears essential to develop a system of professional development in which teachers can enhance their pedagogy through interchange, reflection and refinement of the craft. This will not occur until research and practice are perceived as having supporting roles and there are mechanisms in place to encourage such professional growth.

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THE MATHEMATICS TEACHER AS RESEARCHER IN THE DIAGNOSIS OF CONCEPTUAL UNDERSTANDING

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This paper describes the extension to a teacher education program of a research study designed to assess conceptual understanding of multiplication. The study, which utilized clinical interviewing and supporting instruments, was originally conducted as a research study with university students as subjects and later extended to secondary students using prospective and practicing mathematics teachers in the role of researchers. The teachers received one hour of instruction in clinical interviewing and the use of the instruments. The results obtained by the teacher researchers, who were knowledgeable in the domain of mathematics but could hardly be considered trained interviewers, were comparable with those obtained in the original research study. The design of the instruments elicited sufficiently robust protocols to permit the teachers who used them to diagnose multiplicative understandings in junior and senior high school students.

The original research study sought to determine whether multiplication was held in cognitive structure prototypically or according to characteristic attributes (Schmittau, 1987, 1988). It also questioned whether spontaneous and formal concepts of multiplication had been integrated in the cognitive structure of subjects (Vygotsky, 1962). Instruments to assess prototypicality required subjects to rate instances of multiplication for degree of membership in the category, using a scale of "1" to "7". "1" signified that the instance definitely belonged in the category, while "7" indicated that the instance was a very poor exemplar of the category or did not belong in the category at all.

Subjects were then interviewed about their ratings. They were first asked the question: "What is multiplication? What does it mean to you to multiply?" After responding, they were asked with respect to each instance of multiplication which they had rated: "In what sense do you consider this (i.e., the specific instance under consideration) to be multiplication?" A flexible clinical interview format was followed in probing the responses, in order to assess the conceptual understandings of subjects and to obtain information beyond that of the rating instrument. The subjects' own meanings were elicited, and subjects were consistently directed back to those meanings. Near the end of the interview subjects were given the opportunity to revise their original ratings.

The prospective and practicing teachers were provided with one hour of instruction in the interviewing process which emphasized the difference between clinical interviewing and Socratic teaching, and stressed such aspects of the interviewing process as putting subjects at ease, informing them that their own meanings were of interest rather than the correctness of their responses, and probing beneath surface responses to elicit deeper meanings. In utilizing the interviewing process and the supporting instruments for diagnostic purposes, the teachers became aware of the need to overcome certain psychological effects of formal instruction, including the tendency to respond out of a psychological set and the predisposition to rote responses to which the students had become accustomed to expect acceptance.

In addition, the teachers found evidence among secondary students of the erosion of autonomy which characterized thirty percent of the original subjects, manifested in their refusal to change their ratings from "1" even when it became obvious to them that the instance so rated had no meaning for them. Bowing to mathematical authority, they held that they had never trusted their own meanings.

The table below presents the mean ratings for all instances for both the original study and its extension. Ten university students comprised the sample for the original research. Fifteen secondary students participated in the extension, which assigned each of fifteen practicing or prospective secondary mathematics teachers to a different secondary school student.

Mean Ratings

	Research		*Extension*	
Instance	Original	Revised	Original	Revised
4 x 3	1	1	1	1
2/3 x 4/5	1.5	2.3	2.1	2.3
ab	1.2	1.9	1.4	1.7
(2x + y)(x + 3y)	1.2	2.7	2.3	2.8
(-5) x 2	1.2	2.2	1.7	1.9
(-3) (-2)	1.2	2.6	1.3	2.2
$\sqrt{2\pi}$	2.2	3.4	2.7	2.8
A = bh	1.4	2.3	2.2	2.3
Mean	1.4	2.3	1.8	2.1

As in previous studies (Rosch,1983, 1973; Armstrong, Gleitman, & Gleitman, 1983), graded responses were accepted as evidence of prototypicality. In both the research study and the extension the data revealed that multiplication was organized around a single instance, "4 x 3", which functioned as a model for multiplication across polynomial and numeric domains, and to which meaning across these domains was consistently referred. The conceptual structure of the model instance was determined and expressed in propositional form as the repeated addition of generally small positive integers. Corroborating data from subjects' verbal responses to the question concerning the meaning of multiplication confirmed the structure of the model instance, as all subjects' meanings for multiplication could be summarized as follows. Multiplication was viewed as a short form of addition which enabled subjects to determine the total number of items in n groups of m objects each, where m and n were positive integers. Alternatives to multiplication included adding $m + m + m + \ldots + m$ (n times) or counting the total number of objects across the n groups. This concept of multiplication was reported by all the secondary students as well.

Data from the original research study revealed that the tendency toward conceptual integration cut across all of the following lines of diversity: gender, major field, previous mathematics background, and proximity to the last formal mathematics instruction. Similarly, in the investigation with secondary students, the teacher researchers were surprised to discover that even some high achieving students possessed procedural competence but little understanding of the mathematics concepts involved.

Fraction multiplication, for example, was not well understood by either the college subjects or the high school students. Neither the pie diagrams popular in elementary school textbooks nor the number line were helpful to subjects in mediating meaning. Further, the high school students, like their college counterparts, generally found no meaning in the product of two negative numbers. They could not envision a situation in which two numbers which possessed both magnitude and direction could meaningfully form a product. In the case of irrationals, it was not only the product but the numbers themselves which were not well understood. Inadequate conceptualizations of area were found in about half the subjects in both groups. And the pervasive employment in high school mathematics of the so-called "FOIL" method for binomial multiplication was found to be obscure rather than promote an understanding of polynomial multiplication.

As a consequence of replicating the research study, the prospective and practicing teachers began to realize the early level of mathematical preparation at which failure to understand the requisite concepts had originated. The result was a realization of why remediation could not be accomplished by merely "correcting the procedure", but required an understanding of the conceptual structure of the mathematics in question, as well as a determination of the student's understandings, and an adequate approach to the reconceptualization required.

In general, the ratings of the high school students do not corroborate their verbal data to the extent found in the original research study, and they reflect a greater reluctance to revise ratings following the interview. These discrepancies seem to center around the failure on the part of the teachers to probe as consistently as would trained interviewers, resulting in ratings somewhat lower (i.e., reflecting greater meaning) than the verbal data would suggest. Secondary students who did revise their ratings actually lowered them in a few instances; none of the college subjects lowered ratings for any of the instances. The teachers were sensitive to these discrepancies, however. In addition, it is important to note that the original study required subjects to make concept maps, which provided important corroborative data, particularly in instances where interpretation was problematic. The original study was "streamlined", however, to accommodate the exigencies of secondary school teaching. The teachers all had at least an undergraduate degree in mathematics, but were not trained researchers; each had simply selected a high school student for participation with no particular design for the selection process. (Interestingly, the teachers tended to choose high achieving students.) Each teacher's task was to assess the meaning for this single student of a single concept, viz., multiplication, after exposure by the student to eight to twelve years of school mathematics instruction which had addressed the concept in a variety of numeric, algebraic and geometric contexts. Results were then pooled and discussed, together with implications for instruction.

Certain aspects of the original study of interest to researchers, such as the persistence of featural organization after years of instruction along formal definitional lines, and the implications of the findings for the development of early number concepts, were not explored with the teachers.

The replication served to highlight for the teachers the difference between procedural and conceptual knowledge, however, and in the interview process they acquired a valuable diagnostic tool. In addition, some new information surfaced in the replication in the way of responses on the part of the high school students which added to the research data in an interesting and productive way. As noted above, in both the original study and the extension the exemplar appeared to function in cognitive structure as a spontaneous concept to which the formal mathematical or algorithmic structures were often linked with difficulty or not at all. Where such linkages had not occurred, the algorithms for the products appeared to have been learned by rote. Subjects reported that these algorithms had no meaning and they often could not remember them or apply them accurately. One secondary student, however, suggested that perhaps his definition for multiplication was inadequate or that multiple definitions for multiplication were required.

In summary, while the prospective and practicing teachers were unable to make the fine discriminations characteristic of trained researchers, the project demonstrated that a research study designed to provide robust protocols can be successfully utilized by teachers untrained in research to elicit valuable information about students' mathematical understandings.

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